We want to define a logarithm as in calculus as the inverse function of the exponential. We have seen that if \( z = x + iy \) then the definition

\[
e^z = e^x \cos y + ie^x \sin y
\]

has the properties that we wish for an exponential. That is

\[
e^{z+w} = e^z e^w
\]

and \( e^0 = 1 \). Furthermore with this definition the function is entire (analytic everywhere) and we have \( \frac{d}{dz} e^z = e^z \).

We note that the values of \( e^x \) for real \( x \) are real and strictly positive and that the set of all values is the set \((0, \infty)\) and every positive real number has exactly one preimage. This implies that we can define \( \ln \) the natural logarithm with domain \((0, \infty)\) by rule \( \ln x = y \) means \( x = e^y \). That is

\[
\ln(e^y) = y, e^{\ln(x)} = x.
\]

We also observed that the chain rule implies that

\[
1 = \frac{d}{dx} x = e^{\ln x} \frac{d}{dx} \ln x = x \frac{d}{dx} \ln x
\]

so

\[
\frac{d}{dx} \ln x = \frac{1}{x}
\]

for \( x > 0 \). (Here we also needed the inverse function theorem.)

We want to do something similar for the complex exponential. We immediately run into a problem with this. We note that if \( z = x + iy \) and \( z \neq 0 \) then

\[
z = (x^2 + y^2)^{\frac{1}{2}} \left( \frac{x}{(x^2 + y^2)^{\frac{1}{2}}} + i \frac{y}{(x^2 + y^2)^{\frac{1}{2}}} \right).
\]

Since \( (x^2 + y^2)^{\frac{1}{2}} > 0 \) and \( \frac{x}{(x^2 + y^2)^{\frac{1}{2}}} + i \frac{y}{(x^2 + y^2)^{\frac{1}{2}}} \) is on the unit circle we see that

\[
\frac{x}{(x^2 + y^2)^{\frac{1}{2}}} + i \frac{y}{(x^2 + y^2)^{\frac{1}{2}}} = \cos \theta + i \sin \theta.
\]

So if we set \( u = \ln((x^2 + y^2)^{\frac{1}{2}}) \) then \( z = e^{u+i\theta} \). Thus if \( z \neq 0 \) there is a complex number \( w \) such that \( z = e^w \). We also note that the values of the exponential function are all non-zero. The only problem is that there are many choices of
Indeed, if \( e^a = e^b \) with \( a, b \) complex numbers then \( e^{a-b} = 1 \). This implies \( a - b = i\theta \) and \( \cos \theta = 1 \) and \( \sin \theta = 0 \). This implies that \( \theta = 2\pi k \) with \( k \) an integer. Conversely, \( e^{2\pi ik} \) with \( k \) an integer is 1. We conclude that the set of all \( w \) with \( e^w = z \) is \( \{ \ln((x^2 + y^2)^{\frac{1}{2}}) + i\theta | k = 0, \pm 1, \pm 2, \ldots \} \). The book uses the notation \( \log z \) for this set.

We have seen that one way to drop this ambiguity is to use the principle branch of the argument. That is, constrain \( \theta \) to satisfy \( -\pi < \theta \leq \pi \). With this constraint we will use the notation (as in the book)

\[
\log z = \ln((x^2 + y^2)^{\frac{1}{2}}) + i\theta = \ln((x^2 + y^2)^{\frac{1}{2}}) + i\text{Arg}(z).
\]

We note that we can write this formula out in three different ways in three overlapping domains \( x > 0, y > 0 \) and \( y < 0 \) and we have

\[
\ln((x^2 + y^2)^{\frac{1}{2}}) + i\arcsin \frac{y}{(x^2 + y^2)^{\frac{1}{2}}} \text{ if } x > 0
\]

here \( \arcsin \) is normalized to take values between \( -\frac{\pi}{2} \) and \( \frac{\pi}{2} \). If \( y > 0 \) then we take

\[
\ln((x^2 + y^2)^{\frac{1}{2}}) + i\arccos \frac{x}{(x^2 + y^2)^{\frac{1}{2}}}
\]

here \( \arccos \) takes values between 0 and \( \pi \). Finally if \( y < 0 \) we take

\[
\ln((x^2 + y^2)^{\frac{1}{2}}) - i\arccos \frac{x}{(x^2 + y^2)^{\frac{1}{2}}}
\]

we note that the only points not covered by these three sets are the points with \( y = 0 \) and \( x \leq 0 \). In other words we must not use the value \( \pi \) of \( \text{Arg} \).

Our next task is to see that \( \log \) is analytic in the domain \( x + iy \) with \( y \neq 0 \) or \( y = 0 \) and \( x > 0 \). That is on the set \( \mathbb{C} - (-\infty, 0] \). It is enough to check that it is analytic in each of the indicated domains: \( x > 0, y > 0, y < 0 \). In all cases

\[
u(x, y) = \ln((x^2 + y^2)^{\frac{1}{2}})
\]

so

\[
u_x(x, y) = \frac{x}{x^2 + y^2}, u_y(x, y) = \frac{y}{x^2 + y^2}.
\]

Our problem is to calculate \( v_x \) and \( v_y \) in each of the above cases and show that they are continuous and satisfy the Cauchy Riemann equations. We note that in the indicated domains

\[
\arcsin'(x) = \frac{1}{\sqrt{1-x^2}}, \arccos'(x) = -\frac{1}{\sqrt{1-x^2}}
\]
If \( x > 0 \) then \( v(x, y) = \arcsin \frac{y}{(x^2 + y^2)^{\frac{1}{2}}} \)

\[
v_x(x, y) = \frac{\partial}{\partial x} \left( \arcsin \frac{y}{(x^2 + y^2)^{\frac{1}{2}}} \right) = -\frac{1}{\sqrt{1 - \left( \frac{y}{(x^2 + y^2)^{\frac{1}{2}}} \right)^2}} \left( \frac{-xy}{(x^2 + y^2)^{\frac{1}{2}}} \right) = \frac{1}{\sqrt{1 - \frac{y^2}{x^2 + y^2}}} \left( \frac{-xy}{(x^2 + y^2)^{\frac{1}{2}}} \right)
\]

So if \( x > 0 \) then

\[
v_x(x, y) = -\frac{y}{x^2 + y^2}.
\]

Similarly (we are still assuming that \( x > 0 \))

\[
v_y(x, y) = \frac{\partial}{\partial y} \left( \arcsin \frac{y}{(x^2 + y^2)^{\frac{1}{2}}} \right) = -\frac{1}{\sqrt{1 - \left( \frac{y}{(x^2 + y^2)^{\frac{1}{2}}} \right)^2}} \left( \frac{(x^2 + y^2)^{\frac{1}{2}} - \frac{y^2}{(x^2 + y^2)^{\frac{1}{2}}}}{x^2 + y^2} \right) \frac{x}{x^2 + y^2} = \frac{x}{x^2 + y^2}.
\]

Thus the Cauchy Riemann equations are satisfied and since these partials are continuous for \( x > 0 \) the function \( \text{Log}(z) \) is analytic for \( \text{Re} \ z > 0 \).

We now consider the formula above for \( y > 0 \). This time (here the algebraic manipulation is the same as the case above with the role of \( x \) and \( y \) interchanged)

\[
v_x(x, y) = \frac{\partial}{\partial x} \left( \arccos \frac{x}{(x^2 + y^2)^{\frac{1}{2}}} \right) = -\frac{1}{\sqrt{1 - \left( \frac{x}{(x^2 + y^2)^{\frac{1}{2}}} \right)^2}} \left( \frac{(x^2 + y^2)^{\frac{1}{2}} - \frac{x^2}{(x^2 + y^2)^{\frac{1}{2}}}}{x^2 + y^2} \right) \frac{x}{x^2 + y^2} = -\frac{\sqrt{x^2 + y^2}}{y} \left( \frac{(x^2 + y^2)^{\frac{1}{2}} - \frac{x^2}{(x^2 + y^2)^{\frac{1}{2}}}}{x^2 + y^2} \right)
\]
\[
- \frac{1}{y} \frac{y^2}{x^2 + y^2} = - \frac{y}{x^2 + y^2}.
\]

and
\[
v_y(x, y) = \frac{\partial}{\partial y} \left( \arccos \frac{x}{(x^2 + y^2)^{\frac{1}{2}}} \right) = - \frac{\sqrt{x^2 + y^2}}{y} \frac{(-xy)}{(x^2 + y^2)^{\frac{3}{2}}} = \frac{x}{x^2 + y^2}.
\]

So again the Cauchy-Riemann equations are satisfied.

Finally we look at the case when \( y < 0 \). Here there is a slight modification.

Consider the expression
\[
\frac{1}{\sqrt{1 - \frac{x^2}{x^2 + y^2}}} = \frac{1}{\sqrt{\frac{y^2}{x^2 + y^2}}} = \frac{\sqrt{x^2 + y^2}}{|y|}.
\]

This last expression is
\[
- \frac{\sqrt{x^2 + y^2}}{y}
\]

if \( y < 0 \). Now in the set \( y < 0 \) we have
\[
v(x, y) = - \arccos \left( \frac{x}{\sqrt{x^2 + y^2}} \right)
\]

so
\[
v_x(x, y) = - \left( - \frac{1}{\sqrt{1 - \frac{x^2}{x^2 + y^2}}} \left( \frac{(x^2 + y^2)^{\frac{1}{2}} - \frac{x^2}{(x^2 + y^2)^{\frac{3}{2}}}}{x^2 + y^2} \right) \right)
\]
\[
\frac{1}{\sqrt{1 - \frac{x^2}{x^2 + y^2}}} \left( \frac{(x^2 + y^2)^{\frac{1}{2}} - \frac{x^2}{(x^2 + y^2)^{\frac{3}{2}}}}{x^2 + y^2} \right) = - \frac{\sqrt{x^2 + y^2}}{y} \left( \frac{(x^2 + y^2)^{\frac{1}{2}} - \frac{x^2}{(x^2 + y^2)^{\frac{3}{2}}}}{x^2 + y^2} \right)
\]
\[
= - \frac{y}{x^2 + y^2}.
\]

In the last step we used the calculation in the previous case \( (y > 0) \). Finally,
\[
v_y(x, y) = \frac{\partial}{\partial y} \left( - \arccos \frac{x}{(x^2 + y^2)^{\frac{1}{2}}} \right) = \]
Thus the Cauchy-Riemann equations are also true in this case.

This gives that on the entire set

\[
\log'(x + iy) = u_x(x, y) - iu_y(x, y)
\]

\[
= \frac{x}{x^2 + y^2} - i \frac{y}{x^2 + y^2} = \frac{x - iy}{x^2 + y^2} = \frac{\bar{z}}{z} = \frac{1}{z}.
\]

Exercises:

1. Carry out all of the details in the last case \((y < 0)\).

2. Prove that the union of the three sets \(\{z \mid \text{Re} z > 0\}, \{z \mid \text{Im} z > 0\}, \{z \mid \text{Im} z < 0\}\) is the set difference

\[
\mathbb{C} - (-\infty, 0].
\]

That is all complex numbers except for the ones in \((-\infty, 0]\).

3. Show that if \(u(x, y) = \ln \left(\sqrt{x^2 + y^2}\right)\) and if \((x, y) \neq (0, 0)\) then

\[
\frac{\partial^2}{\partial x^2} u(x, y) + \frac{\partial^2}{\partial y^2} u(x, y) = 0.
\]

4. Show that if \(v(x, y) = \arcsin \frac{y}{(x^2 + y^2)^{\frac{1}{2}}}\) and \(x > 0\) then

\[
\frac{\partial^2}{\partial x^2} v(x, y) + \frac{\partial^2}{\partial y^2} v(x, y) = 0.
\]

5. Show that if \(v(x, y) = \arccos \frac{x}{(x^2 + y^2)^{\frac{1}{2}}}\) and \(y \neq 0\) then

\[
\frac{\partial^2}{\partial x^2} v(x, y) + \frac{\partial^2}{\partial y^2} v(x, y) = 0.
\]