Power series

A series is the sequence of partial sums of a sequence. Thus if \( \{a_n\} \) is a sequence of complex numbers then the corresponding series is \( \{s_n\} \) with \( s_1 = a_1, s_2 = a_1 + a_2, \ldots, s_n = a_1 + a_2 + \ldots + a_n \). We will denote the series by \( \sum_{n=1}^{\infty} a_n \). We say that the series converges if \( \lim_{n \to \infty} s_n = S \) exists in this case we write

\[
\sum_{n=1}^{\infty} a_n = S.
\]

The sequence could start with \( a_0 \) or \( a_{-1} \) or \( a_{10} \) or any other \( k \). If the indexing of the sequence starts with \( k \) then we write \( \sum_{n=k}^{\infty} a_n \).

Example.
1. Let \( z \) be a complex number and \( a_n = z^n \) then we assert that the series \( \sum_{n=0}^{\infty} z^n \) converges for \( |z| < 1 \) to \( \frac{1}{1-z} \).

   To see this we note that \( s_n = 1 + \ldots + z^n \) for \( n = 1, 2, \ldots \). Thus

\[
s_n = \frac{1 - z^{n+1}}{1-z}
\]

(check with polynomial long division). If \( |z| < 1 \) then \( \lim_{n \to \infty} z^{n+1} = 0 \). So \( \lim_{n \to \infty} s_n = \frac{1}{1-z} \).

2. If \( \sum_{n=1}^{\infty} a_n = S \) and \( \sum_{n=1}^{\infty} b_n = T \) then \( \sum_{n=1}^{\infty} (a_n + b_n) = S + T \).

   This follows from the fact that the limit of a sum is the sum of the limits.

The Cauchy-Criterion which we will now describe is a basic aspect of the structure of the real and complex numbers. In a rigorous development of the real numbers some condition that is equivalent to the assertion in the criterion that a limit exists is assumed as an axiom. We will defer this discussion to a course such as math 140. Here we will just state the criterion as a theorem and prove that convergence implies the criterion.

**Theorem 1** Let \( \{a_n\} \) be a sequence of complex numbers then the sequence converges if and only if given \( \varepsilon > 0 \) there exists \( N \) such that if \( n, m \geq N \) then \( |a_n - a_m| < \varepsilon \).

This assertion says that to check convergence one need only check to see if the terms of the sequence become arbitrarily close as the indices get large. We prove that if \( \lim_{n \to \infty} a_n \) exists then given \( \varepsilon > 0 \) there exists \( N \) so that if \( n, m \geq N \) then \( |a_n - a_m| < \varepsilon \). Let \( A \) be the assumed limit of the sequence.
Let $\eta > 0$ be given. Then there exists $N(\eta)$ such that if $n \geq N(\eta)$ then $|a_n - A| < \eta$. If $n, m \geq N(\eta)$ then

$$|a_n - a_m| = |a_n - A + A - a_m| \leq |a_n - A| + |a_m - A| < 2\eta.$$ 

So take $N = N(\frac{\varepsilon}{2})$.

The next result is critical to the theory of power series and follows from the Cauchy-Criterion (the part that we proved).

**Lemma 2** If the series $\sum_{n=k}^{\infty} a_n$ converges then $\lim_{n \to \infty} |a_n| = 0$.

**Proof.** Given $\varepsilon > 0$ there exists $N$ such that if $n, m \geq N$ then $|s_n - s_m| < \varepsilon$. This implies that if $n > N$ then $|s_n - s_{n-1}| < \varepsilon$. Now $s_n - s_{n-1} = a_n$. So we have shown that given $\varepsilon > 0$ there exists $N$ so that if $n > N$ then $|a_n| < \varepsilon$.

We will now take another look at the example above. We were considering the series $\sum_{n=0}^{\infty} z^n$. We saw that if $|z| < 1$ then the series converges to $\frac{1}{1-z}$. If $|z| > 1$ then the sequence $\{ |z^n| \}$ diverges. If $|z| = 1$ then the sequence has limit 1 not 0. This implies that the series converges if and only if $|z| < 1$.

This is a special case of an important result for power series. In general a power series is a series of the form

$$\sum_{n=0}^{\infty} a_n (z - z_o)^n.$$ 

Here we use the convention that $(z - z_o)^0 = 1$ for all $z, z_o$. We note that if the series converges for some $z$ with $|z - z_o| = r$ and $r > 0$ then we must have

$$\lim_{n \to \infty} |a_n||z - z_o|^n = 0.$$ 

This implies that there exists $N$ such that if $n \geq N$ then $|a_n|r^n < 1$ (here we could replaces 1 with any $\varepsilon > 0$). This implies that if $n \geq N$ then $|a_n| < \frac{1}{r^n}$. Now suppose that $|w - z_o| < s < r$. Then if $n \geq N$ we have

$$|a_n||w - z_o|^n < \frac{1}{r^n} s^n = \left( \frac{s}{r} \right)^n.$$ 

This implies that if $m, n \geq N$ and $m > n$ then

$$\left| \sum_{k=0}^{m} a_k (w - z_o)^k - \sum_{k=0}^{n} a_k (w - z_o)^k \right| = \left| \sum_{k=n}^{m} a_k (w - z_o)^k \right|$$
\[
\leq \sum_{k=n}^{m} |a_k||w - z_o|^k \leq \sum_{k=n}^{m} \left( \frac{s}{r} \right)^k \leq \sum_{k=n}^{\infty} \left( \frac{s}{r} \right)^k = \left( \frac{s}{r} \right)^n \frac{1}{1 - (\frac{s}{r})} \leq \left( \frac{s}{r} \right)^N \frac{1}{1 - (\frac{s}{r})}.
\]

This expression goes to 0 with \( N \). The Cauchy-Criterion implies that the series converges if \( |w - z_o| < s \).

We have proved

**Lemma 3** If the series \( \sum_{n=0}^{\infty} a_n(z - z_o)^n \) converges for one \( z \) with \( |z - z_o| = r \) then it converges for every \( w \) with \( |w - z_o| < r \).

The argument used above also proves

**Lemma 4** Consider the series \( \sum_{n=0}^{\infty} a_n(z - z_o)^n \). Assume that \( r > 0 \), \( C > 0 \) and \( N \) have been found such that if \( n \geq N \) then \( |a_n| < \frac{C}{r^n} \) then the series converges for all \( w \) with \( |w - z_o| < r \).

We set \( R \) equal to the set of all \( r > 0 \) such that the series converges for some \( z \) with \( |z - z_o| = r \). We have shown (in Lemma 3) that if \( r \in R \) then \((0, r) \subset R \). This implies that the set \( R \) is of one of the following types:

1. \( R = \emptyset \).
2. \( (0, R) \) with 0 < \( R < \infty \).
3. \( (0, R] \) with 0 < \( R < \infty \).
4. \( (0, \infty) \).

In the first case the series converges only for \( z = z_o \) and the value is \( a_0 \) and we say that the radius of convergence is 0. In cases 2. and 3. we say that the radius of convergence is \( R \) and in case 4 we say that the radius of convergence is \( \infty \).

**Examples.**

3. \( \sum_{n=0}^{\infty} n!z^n \) has radius of convergence 0 (see Exercise 1).

4. \( \sum_{n=0}^{\infty} z^n \) has radius of convergence 1. This was shown above. We note that the series does not converge for any \( z \) on the unit circle.

5. \( \sum_{n=0}^{\infty} \frac{z^n}{n+1} \) has radius of convergence 1 and converges for \( z = -1 \) (see Exercise 2).
6. \( \sum_{n=0}^{\infty} \frac{z^n}{n!} \) has radius of convergence \( \infty \). To prove this we note that \( n! = 1 \cdot 2 \cdot 3 \cdots (n-1) \cdot n \). Thus if we set \( \left\lfloor \frac{n}{2} \right\rfloor \) equal to \( \frac{n}{2} \) if \( n \) is even and \( \frac{n+1}{2} \) if \( n \) is odd. Then if \( n \geq 2 \) we have

\[
n! \geq \left\lfloor \frac{n}{2} \right\rfloor \cdot \left( \left\lfloor \frac{n}{2} \right\rfloor + 1 \right) \cdots n \geq \left( \left\lfloor \frac{n}{2} \right\rfloor \right)^{n-\left\lfloor \frac{n}{2} \right\rfloor}.
\]

We note that this lower bound is \( \left( \frac{n}{2} \right)^{\frac{n}{2}} \) if \( n \) is even and \( \left( \frac{n}{2} + \frac{1}{2} \right)^{\frac{n}{2} - \frac{1}{2}} \) if \( n \) is odd. In both cases it is at least

\[
\left( \frac{n}{2} \right)^{\frac{n}{2} - \frac{1}{2}}.
\]

We therefore see that if \( n > 2 \) then \( \frac{1}{n!} \leq \frac{1}{\left( \frac{n}{2} \right)^{\frac{n}{2} - \frac{1}{2}}} \). Let \( r > 0 \) be given. If \( N > 2r^2 \) and if \( n \geq N \) then we have

\[
\frac{1}{n!} \leq \frac{1}{\left( \frac{n}{2} \right)^{\frac{n}{2} - 1}} \leq \frac{1}{r^{n-1}} \leq \frac{1}{r^n}.
\]

If we write \( C = r \) then we can apply Lemma 4 and see that the series \( \sum_{n=0}^{\infty} \frac{z^n}{n!} \) converges for \( |z| < r \).

This example is cumbersome. By as similar method we can prove

**Theorem 5** Consider the series \( \sum_{n=0}^{\infty} a_n (z - z_o)^n \). If \( \lim_{n \to \infty} \left( \frac{|a_n|}{|a_{n+1}|} \right) \) exists then the limit is the radius of convergence (here we allow \( \infty \)).

**Proof.** Let \( R = \lim_{n \to \infty} \left( \frac{|a_n|}{|a_{n+1}|} \right) \). If \( R = \infty \) this means that given \( r > 0 \) there exists \( N \) so that if \( n \geq N \) then \( \frac{|a_n|}{|a_{n+1}|} > r \). If \( R < \infty \) this implies that if \( 0 < r < R \) then there exists \( N \) so that if \( n \geq N \) then \( \frac{|a_n|}{|a_{n+1}|} > r \). In either case we have

\[
\frac{|a_{n+1}|}{|a_n|} < \frac{1}{r}.
\]

Thus for \( n = N \) we have

\[
|a_{N+1}| < \frac{|a_N|}{r}
\]

and

\[
|a_{N+2}| < \frac{|a_{N+1}|}{r} < \frac{|a_N|}{r^2}
\]
continuing in this way we have

\[ |a_{N+k}| < \frac{|a_N|}{r^k}. \]

Now we can apply Lemma 4 with \( C = |a_N| r^N \) to see that if \( 0 < s < r \) then
the series converges. This implies that if \( s < R \) then the series converges. So \( R \) is at least the radius of convergence. Thus if \( R = \infty \) the theorem is proved.

Assume \( R < \infty \) we wish to show that if \(|z - z_o| > R\) then the series doesn’t converge. Then if \( r > R \) there exists \( N \) so that of \( n \geq N \) then \( \frac{|a_n|}{|a_{n+1}|} < r \) thus

\[ \frac{|a_{n+1}|}{|a_n|} > \frac{1}{r}. \]

So applying the same argument we have

\[ |a_{N+k}| > \frac{|a_N|}{r^k}. \]

Thus if \(|z - z_o| = s > r\) then

\[ |a_{N+k} (z - z_o)^{N+k}| = |a_{N+k}| s^{N+k} > s^N |a_N| \left( \frac{s}{r} \right)^k. \]

Since \( \frac{s}{r} > 1 \) this implies that \( \lim_{k \to \infty} |a_{N+k} (z - z_o)^{N+k}| = \infty \). So the series diverges. \( \blacksquare \)

**Example.**

6. (Again) \( a_n = \frac{1}{n!} \). So \( \frac{|a_n|}{|a_{n+1}|} = n + 1. \) \( \lim_{n \to \infty} (n + 1) = \infty. \)

**Theorem 6** If the power series \( \sum_{n=0}^{\infty} a_n (z - z_o)^n \) has radius of convergence \( R > 0 \) then power series \( \sum_{n=1}^{\infty} na_n (z - z_o)^{n-1} \) has radius of convergence \( R \).

**Proof.** Suppose that \( r < R \). Then there exists \( N \) so that if \( n \geq N \) then \( |a_n| < \frac{1}{r^n}. \) Let \( 0 < s < r \). Then \( |a_n| < \frac{1}{s^n} \left( \frac{s}{r} \right)^n. \) Since \( \frac{s}{r} < 1 \) we see that there exists \( N_1 > N \) such that \( n \left( \frac{s}{r} \right)^n < 1 \) for \( n > N_1 \). Thus if \( n \geq N_1 \) then \( n |a_n| < \frac{1}{s^n}. \) So Lemma 4 implies that the series \( \sum_{n=0}^{\infty} na_n (z - z_o)^n \) converges for \(|z - z_o| < s\). Since \( r \) is arbitrary subject to the condition that \( r < R \) this implies that this series converges for \( s < R \). The convergence of this series
implies the convergence of the series in the statement. Since if \( z \neq z_0 \) then
\[
\sum_{n=1}^{\infty} na_n(z - z_0)^{n-1} = \frac{1}{z-z_0} \sum_{n=0}^{\infty} na_n(z - z_0)^n.
\]

Now suppose that \( r > R \) and there exists \( z \) with \( |z - z_0| = r \) so that
\[
\sum_{n=0}^{\infty} na_n(z - z_0)^n \text{ converges we show that this is impossible (i.e. it implies something false).}
\]
This implies that there exists \( N > 0 \) so that if \( n \geq N \) then
\[
|a_n| < \frac{1}{r^n} \leq \frac{1}{r^N}.
\]
Thus the original series converges for \( |w - z_0| < s < r \). This includes elements \( s \) with \( s > R \). This is the desired false implication. \( \blacksquare \)

We now come to the crux of the matter

**Theorem 7** If the power series \( \sum_{n=0}^{\infty} a_n(z - z_0)^n \) has radius of convergence \( R > 0 \) then the function \( f(z) = \sum_{n=0}^{\infty} a_n(z - z_0)^n \) is analytic for \( |z - z_0| < R \) with \( f'(z) = \sum_{n=0}^{\infty} na_n(z - z_0)^{n-1} \). Furthermore, \( f \) has derivatives of all orders and \( f^{(k)}(z) = \sum_{n=k}^{\infty} n(n-1) \cdots (n-k+1) a_n(z - z_0)^{n-k} \).

**Proof.** We assume \( 0 < s < r < R \). Then from the definition of the radius of convergence we know that there exists \( N \) such that if \( n \geq N \) then \( |a_n| \leq \frac{1}{r^n} \) and \( n|a_n| < \frac{1}{r^N} \). We now look at \( w, z \in \{u| |u - z_0| < s \} \) and \( w \neq z \). Then
\[
\frac{f(z) - f(w)}{z-w} = \frac{1}{z-w} \left( \sum_{n=0}^{\infty} a_n(z - z_0)^n - \sum_{n=0}^{\infty} a_n(w - z_0)^n \right).
\]
We note that
\[
\sum_{n=0}^{\infty} a_n((z - z_0)^n - (w - z_0)^n) = \sum_{n=0}^{\infty} a_n(z - z_0)^n - \sum_{n=0}^{\infty} a_n(w - z_0)^n
\]
by example 2 above. Thus
\[
\frac{f(z) - f(w)}{z-w} = \sum_{n=0}^{\infty} a_n \left( \frac{(z - z_0)^n - (w - z_0)^n}{z-w} \right).
\]
We now assume that \( m > N \). Then
\[
\sum_{n=0}^{\infty} a_n \left( \frac{(z - z_0)^n - (w - z_0)^n}{z-w} \right) = \sum_{n=0}^{m} a_n \left( \frac{(z - z_0)^n - (w - z_0)^n}{z-w} \right) +
\]
\[
\sum_{n=m+1}^{\infty} a_n \left( \frac{(z - z_0)^n - (w - z_0)^n}{z-w} \right).
\]
\[ \sum_{n=m+1}^{\infty} a_n \left( \frac{(z - z_o)^n - (w - z_o)^n}{z - w} \right). \]

We note that if \( n > 0 \) then
\[
(z - z_o)^n - (w - z_o)^n = (z - z_o)^{n-1} + (z - z_o)^{n-2}(w - z_o) + \ldots + (z - z_o)(w - z_o)^{n-2} + (z - z_o)^{n-1}
\]
(see Exercise 3). Thus
\[
\sum_{n=m+1}^{\infty} a_n \left( \frac{(z - z_o)^n - (w - z_o)^n}{z - w} \right) = \sum_{n=m+1}^{\infty} a_n (z - z_o)^{n-1} + (z - z_o)^{n-2}(w - z_o) + \ldots + (z - z_o)(w - z_o)^{n-2} + (z - z_o)^{n-1}.
\]

This implies that
\[
\left| \sum_{n=m+1}^{\infty} a_n \left( \frac{(z - z_o)^n - (w - z_o)^n}{z - w} \right) \right| \leq \sum_{n=m+1}^{\infty} n|a_n|s^{n-1}
\]
\[
\sum_{n=m+1}^{\infty} a_n (|z - z_o|^n - |w - z_o|^n) \leq \sum_{n=m+1}^{\infty} |a_n|(|z - z_o|^{n-1} + |z - z_o|^{n-2}|w - z_o| + \ldots + |z - z_o||w - z_o|^{n-2} + |z - z_o|^{n-1}) \leq \sum_{n=m+1}^{\infty} n|a_n|s^{n-1}
\]

since \(|z - z_o|^{n-1-b}|w - z_o|^b < s^{n-1-b}s^b = s^{n-1}\) for all \( b = 0, \ldots, n-1 \). We have also chosen \( N \) so that \( n|a_n| < \frac{1}{r^n} \) if \( n \geq N \). Thus since \( n > m > N \) we have
\[
\sum_{n=m+1}^{\infty} n|a_n|s^{n-1} \leq \frac{1}{s} \sum_{n=m+1}^{\infty} \left( \frac{s}{r} \right)^n = \frac{1}{s} \left( \frac{s}{r} \right)^{m+1} \frac{1}{1 - \frac{s}{r}}.
\]

We therefore see that if \( \varepsilon > 0 \) is given then we can choose \( m > N \) such that
\[
\frac{1}{s} \left( \frac{s}{r} \right)^{m+1} \frac{1}{1 - \frac{s}{r}} < \frac{\varepsilon}{3},
\]

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Thus if \( w, z \in \{ u \mid |u - z_o| < s \} \) and \( z \neq w \) then
\[
\left| \sum_{n=m+1}^{\infty} a_n \left( \frac{(z - z_o)^n - (w - z_o)^n}{z - w} \right) \right| < \frac{\varepsilon}{3}
\]

On the other hand for \( m \) fixed at this value we have
\[
\frac{d}{dw} \sum_{n=0}^{m} a_n (w - z_o)^n = \sum_{n=1}^{m} na_n (w - z_o)^{n-1}.
\]

Thus by the definition of the derivative there exists \( \delta > 0 \) such that if \( w, z \in \{ u \mid |u - z_o| < s \} \) and \( 0 < |z - w| < \delta \) then
\[
\left| \sum_{n=0}^{m} a_n \left( \frac{(z - z_o)^n - (w - z_o)^n}{z - w} \right) - \sum_{n=1}^{m} na_n (w - z_o)^{n-1} \right| < \frac{\varepsilon}{3}.
\]

We also note that for the same value of \( m \) we have
\[
\left| \sum_{n=m+1}^{\infty} na_n (w - z_o)^{n-1} \right| < \frac{\varepsilon}{3}.
\]

We can now put everything together
\[
\frac{f(z) - f(w)}{w - z} - \sum_{n=1}^{\infty} na_n (w - z_o)^{n-1} =
\]
\[
\sum_{n=0}^{\infty} a_n \left( \frac{(z - z_o)^n - (w - z_o)^n}{z - w} \right) - \sum_{n=1}^{\infty} na_n (w - z_o)^{n-1} =
\]
\[
\sum_{n=0}^{m} a_n \left( \frac{(z - z_o)^n - (w - z_o)^n}{z - w} \right) - \sum_{n=1}^{m} na_n (w - z_o)^{n-1} +
\]
\[
\sum_{n=m+1}^{\infty} a_n \left( \frac{(z - z_o)^n - (w - z_o)^n}{z - w} \right) - \sum_{n=m+1}^{\infty} na_n (w - z_o)^{n-1}.
\]

Taking absolute values we have
\[
\left| \frac{f(z) - f(w)}{z - w} - \sum_{n=1}^{\infty} na_n (w - z_o)^{n-1} \right| \leq
\]

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\begin{align*}
&\sum_{n=0}^{m} a_n \left( \frac{(z-z_o)^n - (w-z_o)^n}{z-w} \right) - \sum_{n=1}^{m} n a_n (w-z_o)^{n-1} + \\
&\sum_{n=m+1}^{\infty} a_n \left( \frac{(z-z_o)^n - (w-z_o)^n}{z-w} \right) + \sum_{n=1}^{m} n a_n (w-z_o)^{n-1} \\
\end{align*}

each of these terms is less than $\varepsilon$ when $w, z \in \{u \mid |u - z_o| < s\}$ and $0 < |z - w| < \delta$. Thus under these conditions

\[
\left| \frac{f(z) - f(w)}{z-w} - \sum_{n=1}^{\infty} n a_n (w-z_o)^{n-1} \right| < \varepsilon
\]

so

\[
\lim_{z\to w} \frac{f(z) - f(w)}{z-w} = \sum_{n=1}^{\infty} n a_n (w-z_o)^{n-1}.
\]

This proves the first part of the theorem. The formula for the higher derivatives comes by repeated application of the first derivative formula. \(\blacksquare\)

**Examples.**

6. (Yet again). Consider $f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$. Then the radius of convergence of the series is $\infty$. Thus $f$ is entire (analytic on all of $\mathbb{C}$) and the theorem above implies that

\[
f'(z) = \sum_{n=1}^{\infty} n \frac{z^{n-1}}{n!} = \sum_{n=1}^{\infty} \frac{z^{n-1}}{(n-1)!} = \sum_{n=0}^{\infty} \frac{z^n}{n!}.
\]

So $f'(z) = f(z)$. We also note that $f(0) = 1$. We will show in the exercises why if $g(z)$ is analytic on a connected open set containing $0$ and if $g'(z) = g(z), g(0) = 1$ then $g(z) = f(z)$ in its domain. Since $g(z) = e^z$ has this property we see that

\[
e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}.
\]

7. We have

\[
\cosh(z) = \frac{e^z + e^{-z}}{2} = \frac{\sum_{n=0}^{\infty} \frac{z^n}{n!} + \sum_{n=0}^{\infty} (-1)^n \frac{z^n}{n!}}{2}.
\]
Since \(1 + (-1)^n\) is 2 if \(n\) is even and 0 if \(n\) is odd we have

\[
cosh(z) = \sum_{n=0}^{\infty} \frac{z^{2n}}{(2n)!}.
\]

8. Similarly we have

\[
sinh(z) = \frac{e^z - e^{-z}}{2} = \frac{\sum_{n=0}^{\infty} \frac{z^n}{n!} - \sum_{n=0}^{\infty} (-1)^n \frac{z^n}{n!}}{2}
\]

so

\[
sinh(z) = \sum_{n=0}^{\infty} \frac{z^{2n+1}}{(2n+1)!}.
\]

9. We have

\[
\cos(z) = \cosh(iz) = \sum_{n=0}^{\infty} \frac{(iz)^{2n}}{(2n)!}
\]

\[
= \sum_{n=0}^{\infty} (i)^{2n} \frac{z^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!}
\]

and

\[
\sin(z) = -i \sinh(iz) = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!}.
\]

**Taylor Series**

We start with \(f\) an analytic function on the disk \(|z - z_o| < r\). Let \(z\) be a point in the disk and assume \(|z - z_o| < \rho < r\). Then the Cauchy integral theorem implies that

\[
f(z) = \frac{1}{2\pi i} \int_{C_{\rho}} \frac{f(w)}{w-z} \, dw
\]

where \(C_{\rho}\) is the circle \(|w - z_o| = \rho\) oriented counter clockwise. We consider

\[
\frac{1}{w-z} = \frac{1}{(w-z_o) - (z-z_o)} = \frac{1}{w-z_o} \left( \frac{1}{1 - \frac{z-z_o}{w-z_o}} \right)
\]
We assume that $|z - z_o| < s < \rho$

$$\left| \frac{z - z_o}{w - z_o} \right| < \frac{s}{\rho} < 1.$$ 

This implies that

$$\frac{1}{w - z} = \frac{1}{(w - z_o)} \left( \sum_{n=0}^{\infty} \left( \frac{z - z_o}{w - z_o} \right)^n \right).$$

We have for each $m$

$$f(z) = \frac{1}{2\pi i} \int_{C_{\rho}} \frac{f(w)}{w - z} dw =$$

$$\frac{1}{2\pi i} \int_{C_{\rho}} f(w)dw \left( \sum_{n=0}^{m} \left( \frac{z - z_o}{w - z_o} \right)^n \right) + \frac{1}{2\pi i} \int_{C_{\rho}} f(w)dw \left( \sum_{n=m+1}^{\infty} \left( \frac{z - z_o}{w - z_o} \right)^n \right) =$$

$$\sum_{n=0}^{m} \frac{1}{2\pi i} \left( \int_{C_{\rho}} \frac{f(w)dw}{(w - z_o)^{n+1}} \right) (z - z_o)^n +$$

$$\frac{1}{2\pi i} \int_{C_{\rho}} f(w) \left( \sum_{n=m+1}^{\infty} \left( \frac{z - z_o}{w - z_o} \right)^n \right) dw.$$ 

Now

$$\frac{1}{2\pi i} \int_{C_{\rho}} f(w) \left( \sum_{n=m+1}^{\infty} \left( \frac{z - z_o}{w - z_o} \right)^n \right) dw =$$

$$\frac{1}{2\pi} \int_{0}^{2\pi} f(z_o + \rho e^{i\theta}) \left( \sum_{n=m+1}^{\infty} \left( \frac{z - z_o}{\rho e^{i\theta}} \right)^n \right) d\theta.$$ 

Thus

$$\left| \frac{1}{2\pi i} \int_{C_{\rho}} f(w) \left( \sum_{n=m+1}^{\infty} \left( \frac{z - z_o}{w - z_o} \right)^n \right) dw \right| \leq$$

$$\frac{1}{2\pi} \int_{0}^{2\pi} \left| f(z_o + \rho e^{i\theta}) \right| \left| \sum_{n=m+1}^{\infty} \left( \frac{z - z_o}{\rho e^{i\theta}} \right)^n \right| d\theta \leq$$

$$\frac{1}{2\pi} \int_{0}^{2\pi} \left| f(z_o + \rho e^{i\theta}) \right| d\theta \left( \frac{s}{\rho} \right)^{m+1} \frac{1}{1 - \frac{z}{\rho}}$$
since
\[
\left| \sum_{n=m+1}^{\infty} \left( \frac{z - z_0}{\rho e^{i\theta}} \right)^n \right| \leq \sum_{n=m+1}^{\infty} \left| \frac{z - z_0}{\rho e^{i\theta}} \right|^n \\
\leq \sum_{n=m+1}^{\infty} \left( \frac{s}{\rho} \right)^n = \left( \frac{s}{\rho} \right)^{m+1} \frac{1}{1 - \frac{s}{\rho}}.
\]

We therefore have
\[
f(z) - \sum_{n=0}^{m} \frac{1}{2\pi i} \left( \int_{C_{\rho}} \frac{f(w)\,dw}{(w - z_0)^{n+1}} \right) (z - z_0)^n = \\
\frac{1}{2\pi i} \int_{C_{\rho}} \frac{f(w)}{w - z_0} \left( \sum_{n=m+1}^{\infty} \frac{(z - z_0)^n}{w - z_0} \right) \,dw.
\]
So if we set
\[
a_n = \int_{C_{\rho}} \frac{f(w)\,dw}{(w - z_0)^{n+1}}
\]
then
\[
\left| f(z) - \sum_{n=0}^{m} a_n (z - z_0)^n \right| = \\
\left| \frac{1}{2\pi i} \int_{C_{\rho}} \frac{f(w)}{w - z_0} \left( \sum_{n=m+1}^{\infty} \frac{(z - z_0)^n}{w - z_0} \right) \,dw \right| \leq \\
\frac{1}{2\pi} \int_{0}^{2\pi} |f(z_0 + \rho e^{i\theta})| \,d\theta \left( \frac{s}{\rho} \right)^{m+1} \frac{1}{1 - \frac{s}{\rho}}.
\]
Thus the limit
\[
\lim_{m \to \infty} \sum_{n=0}^{m} a_n (z - z_0)^n - f(z) = 0.
\]

We have

**Theorem 8** Let \(f\) be analytic on the disk \(|z - z_0| < r\). Then there exists a power series \(\sum_{n=0}^{\infty} a_n (z - z_0)^n\) with radius of convergence \(R \geq r\) such that

\[
f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n
\]
for $|z - z_0| < r$. Furthermore the series is uniquely determined since $f^{(n)}(z_0) = n!a_n$. Finally

$$a_n = \frac{1}{2\pi i} \int_{C_\rho} \frac{f(w)dw}{(w - z_0)^{n+1}}.$$  

**Proof.** We have proved everything but the formula for the derivative. But this is gotten by taking the formula in Theorem 7 and evaluating it at $z_0$. ■

**Examples.**

The examples 6-9 above give the Taylor series of all of the basic elementary functions.

**Exercises.**

1. Prove that the power series in example 3 has radius of convergence 0. (Hint: Use Theorem 5.)

2. If $a, b$ are complex numbers with $a \neq b$ prove that

$$\frac{a^n - b^n}{a - b} = a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \ldots + ab^{n-2} + b^{n-1}.$$  

Hint:

$$\frac{a^n - b^n}{a - b} = a^{n-1} \frac{1 - \left(\frac{b}{a}\right)^n}{1 - \left(\frac{b}{a}\right)}.$$  

3. Prove the assertion in example 5.

4. Carry out the details in example 9 for $\sin(z)$.

5. Let $f(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$. Since $f(0) = 1$ there exists $r > 0$ so that $f(w) \neq 0$ if $|w| < r$.

a) Use this to show that of $g(z)$ is analytic for $|z| < r$ such that $g'(z) = g(z)$ and $g(0) = 1$ then $g(z) = f(z)$ for $|z| < r$. (Hint: Calculate the derivative of $\frac{g(z)}{f(z)}$.)

b) Show that if $|z| < r$ and $|w| < r$ then $f(z + w) = f(z)f(w)$ (Hint: Let $g(z) = f(w)^{-1}f(z + w)$ and use part (a).

c) Prove that $f(z) \neq 0$ for all $z$, (Hint: Using (b) shown that if $f(z) \neq 0$ for $|z| < r$ and if $|w| < 2r$ then $f(w) = f(w + \frac{w}{2})f(\frac{w}{2})$ which is not 0 since $|\frac{w}{2}| < r$.)