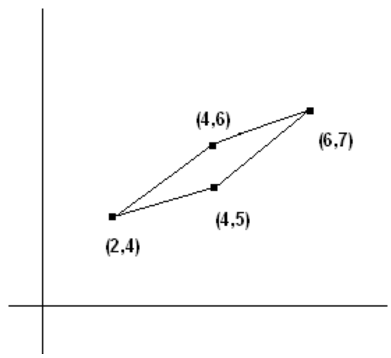
Midterm Exam Solution Set
Math 20e, Fall 2004

1, Draw a picture of the parallelogram in the xy plane with vertices

$$(2,4),(4,6),(4,5),(6,7).$$

Calculate its area.

Solution:



We note that the adjacent sides at the vertex $(2,4)$ have displacement vectors $2\mathbf{i}+2\mathbf{j}$ ($(2,4)$ to $(4,6)$) and $2\mathbf{i}+3\mathbf{j}$ ($(2,4)$ to $(4,5)$). Thus the area is the norm of

$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & 0 \\ 2 & 3 & 0 \end{vmatrix} = 2\mathbf{k}.$$

So the answer is 2.

2. Which of the following two limits exist? If the limit exists calculate it. In either case you must justify your answer.

a) $\lim_{(x,y,z)\to(0,0,0)} \frac{xy+z^3}{1+x^2+y^2+z^2}$.

b) $\lim_{(x,y,z)\to(0,0,0)} \frac{xy+z^3}{x^2+y^2+z^2}$.

Solution:

a) The limit exists since the function is a ratio of two polynomials and the denominator is not 0 at $(0,0,0)$ this implies the limit is the value at $(0,0,0)$ which is 0.

b) If we look at the line $x = y$ and $z = 0$ then the value of the function is $\frac{x^2}{2x^2} = \frac{1}{2}$. If we look at the line $y = z = 0$ then the value is $\frac{0}{x^2} = 0$. Thus the limit does not exist.

3) Assume that f is a function on $\mathbb{R}^3$ that is differentiable at $(1,1,-1)$ and

$$\frac{\partial f}{\partial x}(1,1,-1) = -1, \frac{\partial f}{\partial y}(1,1,-1) = 2, \frac{\partial f}{\partial z}(1,1,-1) = -1.$$

Suppose that $u(t) = \cos t, v(t) = 1 + t, w(t) = -e^t$. Calculate the derivative of $f(u(t), v(t), w(t))$ in t at $t = 0$.

Solution:

$u(0) = 1, v(0) = 2$ and $w(0) = -1$. $u'(0) = \sin(0) = 0, v'(0) = -1, w'(0) = -1$ thus if $g(t) = f(u(t), v(t), w(t))$ then

$$g'(0) = \frac{\partial f}{\partial x}(u(0), v(0), w(0))u'(0) + \frac{\partial f}{\partial y}(u(0), v(0), w(0))v'(0) + \frac{\partial f}{\partial z}(u(0), v(0), w(0))w'(0)$$

$$= (-1)0 + 2(-1) + (-1)(-1) = -1.$$

4. Calculate the arc length of the following two paths curves.

a) $(t, t^2, \frac{2t^3}{3})$, $0 \leq t \leq 1$.

b) $(\cos|t|, \sin|t|, t)$, $-\pi \leq t \leq \pi$. (Hint: You will have to divide the path into two pieces.)

Solutions:

a) The velocity vector is $(1, 2t, t^2)$ thus the arclength is

$$\int_0^1 \|(1, 2t, 2t^2)\| dt = \int_0^1 \sqrt{1 + 4t^2 + 4t^4}\, dt$$

$$= \int_0^1 (1 + 2t^2) dt = 1 + \frac{2}{3} = \frac{5}{3}.$$

b) You must calculate the length of the path $(\cos(-t), \sin(-t), t)$ on the interval $-\pi \leq t \leq 0$ and add it to the length of the path $(\cos t, \sin t, t)$ on the interval $0 \leq t \leq \pi$. The first is

$$\int_{-\pi}^0 \sqrt{\sin^2(-t) + \cos^2(-t) + 1}\, dt = \sqrt{2} \int_{-\pi}^0 dt = \sqrt{2}\,\pi.$$

The second is

$$\int_0^\pi \sqrt{\sin^2 t + \cos^2 t + 1}\, dt = \sqrt{2} \int_0^\pi dt = \sqrt{2}\,\pi.$$

So the answer is $\sqrt{2}\,\pi + \sqrt{2}\,\pi = 2\sqrt{2}\,\pi$.