Solutions for the practice midterm
Math 20e, Fall 2004

1.a) Suppose that **a** and **b** are both non-zero vectors and $\mathbf{a}\cdot\mathbf{b} = 0$ and suppose that c is a scalar. Consider the following equations with x an unknown vector:

$$\mathbf{a}\cdot\mathbf{x} = c$$
$$\mathbf{a}\times\mathbf{x} = \mathbf{b}.$$

Let θ be the angle between **a** and a solution **x**. Calculate $\cos\theta$ and $\sin\theta$.

b) If in part a) $\mathbf{a} = (\sqrt{2},1,1), \mathbf{b} = \sqrt{2}(\mathbf{j}-\mathbf{k}), c = 2$ what is the angle between a solution **x** and **a**? Find all solutions.

Solution:

a) We have $\|\mathbf{a}\|\|\mathbf{x}\|\cos\theta = c$ and $\|\mathbf{a}\|\|\mathbf{x}\|\sin\theta = \|b\|$ thus

$$\|\mathbf{a}\|^2\|\mathbf{x}\|^2\cos^2\theta + \|\mathbf{a}\|^2\|\mathbf{x}\|^2\sin^2\theta = c^2 + \|\mathbf{b}\|^2.$$

This says that

$$\|\mathbf{x}\| = \frac{\sqrt{c^2+\|\mathbf{b}\|^2}}{\|\mathbf{a}\|}.$$

If we substitute this value for $\|x\|$ then we have

$$\cos\theta = \frac{c}{\sqrt{c^2+\|\mathbf{b}\|^2}}, \sin\theta = \frac{\|\mathbf{b}\|}{\sqrt{c^2+\|\mathbf{b}\|^2}}.$$

b) In this case $\|\mathbf{b}\| = 2$ so $\cos\theta = \frac{2}{\sqrt{4+4}} = \frac{1}{\sqrt{2}}, \sin\theta = \frac{2}{\sqrt{4+4}} = \frac{1}{\sqrt{2}}$. Thus $\theta = \frac{\pi}{4}$. We note that for any solution **x**x is orthogonal to $\sqrt{2}(\mathbf{j}-\mathbf{k})$. So **a** and **x** are position vectors to points in the same plane and we know the angle between them and the length of **x**. Thus we know **x**. This implies that **x** is unique.

2. Consider the cone $z = \sqrt{x^2+y^2}$. Calculate the tangent plane at every point on the surface except $(0,0,0)$. Show that the line through the origin in the direction of the position vector (x,y,z) to a point on the surface is completely contained in the tangent plane at the point.

Solution: $\frac{\partial z}{\partial x} = \frac{x}{\sqrt{x^2+y^2}}, \frac{\partial z}{\partial x} = \frac{y}{\sqrt{x^2+y^2}}$ which are both continuous for $(x,y) \neq (0,0)$.

Thus the tangent plane at $(x_o, y_o, \sqrt{x_o^2+y_o^2})$ is given by

$$z - \sqrt{x_o^2+y_o^2} = \frac{x_o}{\sqrt{x_o^2+y_o^2}}(x-x_o) + \frac{y_o}{\sqrt{x_o^2+y_o^2}}(y-y_o).$$

The line through the origin in the direction of $(x_o, y_o, \sqrt{x_o^2+y_o^2})$ is given parimetrically by

$$(x(t),y(t),z(t)) = t(x_o,y_o,\sqrt{x_o^2+y_o^2}\,).$$

Subsituting these values of x,y,z into the equation of the plane we must see that

$$t\sqrt{x_o^2+y_o^2} - \sqrt{x_o^2+y_o^2} = \frac{x_o}{\sqrt{x_o^2+y_o^2}}(tx_o-x_o) + \frac{y_o}{\sqrt{x_o^2+y_o^2}}(ty_o-y_o).$$

The left side of this equation is

$$(t-1)\sqrt{x_o^2+y_o^2}\,.$$

The right side is

$$\frac{(t-1)x_o^2}{\sqrt{x_o^2+y_o^2}} + \frac{(t-1)y_o^2}{\sqrt{x_o^2+y_o^2}} = (t-1)\sqrt{x_o^2+y_o^2}$$

as desired.

3. A function $f(x,y,z)$ is differentiable at $(1,1,-1)$ with gradient $(2,1,1)$ at $(1,1,-1)$. Suppose that $u(t),v(t),w(t)$ are functions of t that are differentiable at $t=1$, $u(1)=v(1)=1$ and $w(1)=-1$ and $u'(1)=-1, v'(1)=2,\ w'(1)=3$. Calculate the derivative of $f(u(t),v(t),w(t))$ at $t=1$.

Solution:

$$\frac{df(u(t),v(t),w(t))}{dt}_{|t=1} = \frac{\partial f}{\partial x}(u(1),v(1),w(1))u'(1) + \frac{\partial f}{\partial y}(u(1),v(1),w(1))v'(1) +$$

$$\frac{\partial f}{\partial z}(u(1),v(1),w(1))w'(1) = \frac{\partial f}{\partial x}(1,1,-1))(-1) + \frac{\partial f}{\partial y}(1,1,-1))2 + \frac{\partial f}{\partial z}(1,1,-1)3 =$$

$$2(-1)+2+3 = 3.$$

4. Calculate the arclength of the path

$$\mathbf{x}(t) = (\cos(t^2),\sin(t^2),t^2)$$

for $0 \le t \le \pi$.

Solution:

$\mathbf{x}'(t) = (-2t\sin(t^2), 2t\cos(t^2).2t)$. Thus

$$\|\mathbf{x}'(t)\| = \sqrt{4t^2(\sin(t^2))^2 + 4t^2\cos(t^2))^2 + 4t^2} =$$

$$\sqrt{4t^2+4t^2} = |t|2\sqrt{2}\,.$$

The arclength is thus the integral $\int_0^\pi \|\mathbf{x}'(t)\|dt = 2\sqrt{2}\int_0^\pi tdt = \sqrt{2}\,\pi^2$.

Alternate solution the arclength is independent of parametrization. This path is a reparametrization of the path $\mathbf{y}(s) = (\cos s, \sin s, s)$ with s replaced by t^2 so $0 \le s \le \pi^2$. Now

$$\mathbf{y}'(s) = (-\sin s, \cos s, 1).$$

So, $\|\mathbf{y}'(s)\| = \sqrt{\sin^2 s + \cos^2 s + 1} = \sqrt{2}$. So the arclength is

$$\int_0^{\pi^2} \sqrt{2}\, dt = \sqrt{2}\,\pi^2.$$