Practice Quizzes II

Practice Quiz 1:

Let $R = \{(x,y)||x-1| < 1, |y-1| < 1\}$. Suppose that f is a real valued function on R that has continuous first and second partial derivatives. Assume that $f(1,1) = -1$ and-1

$$\begin{aligned} \frac{\partial f}{\partial x}(1,1) &= 1, \\ \frac{\partial f}{\partial y}(1,1) &= 2. \end{aligned}$$

Also assume that if (x,y) are in R then

$$\left|\frac{\partial^2 f}{\partial x^2}(x,y)\right| \le 1, \left|\frac{\partial^2 f}{\partial x \partial y}(x,y)\right| \le 1, \left|\frac{\partial^2 f}{\partial y^2}(x,y)\right| \le 1.$$

Calculate the first Taylor polynomial, p_1, of f at $(1,1)$. Use this to calculate an approximate value for $f(1.1, 1.1)$ and give an estimate for the accuracy of this approximation.

Solution: We have

$$p_1(1+h_1, 1+h_2) = -1 + \frac{\partial f}{\partial x}(1,1)h_1 + \frac{\partial f}{\partial y}(1,1)h_2.$$

The Lagrange formula for the remainder is

$$R_1(\mathbf{h}) = \frac{1}{2}\left(\frac{\partial^2 f}{\partial x^2}(\mathbf{z})h_1^2 + 2\frac{\partial^2 f}{\partial x \partial y}(\mathbf{z})h_1 h_2 + \frac{\partial^2 f}{\partial y^2}(\mathbf{z})h_2^2\right)$$

with $\mathbf{z}$ on the line segment joining $(1,1)$ to $(1+h_1, 1+h_2)$. If $(1+h_1, 1+h_2)$ is in R then the entire line segment is in R so we can estimate

$$\begin{aligned} |R_1(\mathbf{h})| &\le \frac{1}{2}\left(\left|\frac{\partial^2 f}{\partial x^2}(\mathbf{z})\right| h_1^2 + 2\left|\frac{\partial^2 f}{\partial x \partial y}\right|(\mathbf{z})|h_1||h_2| + \left|\frac{\partial^2 f}{\partial y^2}\right|(\mathbf{z})h_2^2\right) \\ &\le \frac{1}{2}(h_1^2 + 2|h_1||h_2| + h_2^2). \end{aligned}$$

Thus if $h_1 = h_2 = 0.1$ then $p_1(1+h_1, 1+h_2) = -1 + 0.1 + 0.2 = -0.7$ and an estimate of the error of this approximation is $\frac{1}{2}(4(0.01)) = 0.02$.

Practice Quiz 2:

Define $\beta(x)$ to be given by $\beta(x) = \sin(x)$ for $0 \le x \le \frac{\pi}{4}$ and $\beta(x) = \cos x$ for $\frac{\pi}{4} \le x \le \frac{\pi}{2}$. Draw the domain

$$D = \{(x,y)|0 \le x \le \frac{\pi}{2}, 0 \le y \le \alpha(x)\}.$$

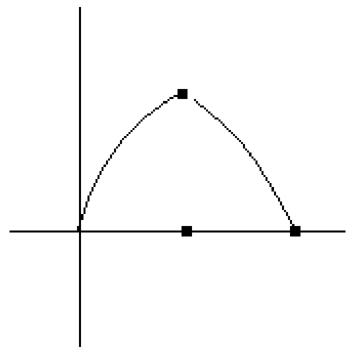

Figure 1:

Calculate

$$\iint\limits_D \sin x dA.$$

Hint: $\sin^2 x = \frac{1-\cos 2x}{2}$.

Solution:

The domain is the area under the graph

Where the plot points are

$$(\frac{\pi}{4}, 0), (\frac{\pi}{2}, 0), (\frac{\pi}{4}, \frac{1}{\sqrt{2}}).$$

The integral is

$$\begin{aligned}
\int_0^{\frac{\pi}{2}} \int_0^{\beta(x)} \sin x dy dx &= \int_0^{\frac{\pi}{4}} \int_0^{\sin x} \sin x dy dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \int_0^{\cos x} \sin x dy dx = \\
\int_0^{\frac{\pi}{4}} \sin^2 x dx + \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin x \cos x dx &= \frac{1}{2} \int_0^{\frac{\pi}{4}} (1 - \cos 2x) dx + \frac{1}{2} \sin^2 x |_{\frac{\pi}{4}}^{\frac{\pi}{2}} = \\
\frac{\pi}{8} - \frac{\sin 2x}{4} |_0^{\frac{\pi}{4}} + \frac{1}{4} &= \frac{\pi}{8}.
\end{aligned}$$