Some Practice Problems involving Green's, Stokes', Gauss' theorems.

1. Let $\mathbf{x}(t) = (a\cos t^2, b\sin t^2)$ with $a, b > 0$ for $0 \le t \le \sqrt{2\pi}$ Calculate $\int_{\mathbf{x}} x dy$. Hint: $\cos^2 t = \frac{1+\cos 2t}{2}$.

2. Let $\mathbf{F} = \frac{-y\mathbf{i}+x\mathbf{j}}{x^2+y^2}$

a) Use Green's theorem to explain why

$$\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = 0$$

if $\mathbf{x}$ is the boundary of a domain that doesn't contain 0.

b) Let $\mathbf{x}(t) = (\cos t, 3\sin t)$, $0 \le t \le 2\pi$. and $\mathbf{F} = \frac{-y\mathbf{i}+x\mathbf{j}}{x^2+y^2}$. Calculate $\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s}$. Hint: Consider the domain between x and the circle $\mathbf{y}(t) = (\cos t, \sin t)$. Use part a) to see that $\int_{\mathbf{x}} \mathbf{F} \cdot d\mathbf{s} = \int_{\mathbf{y}} \mathbf{F} \cdot d\mathbf{s}$.

3. Which if the following vector fields is of the form ∇f? If it is compute an f.

a) $\mathbf{F} = x^2\mathbf{i} + xy\mathbf{j}$.

b) $\mathbf{F} = x^2\mathbf{i} - y^2\mathbf{j}$

c) $\mathbf{F} = y\mathbf{i} - x\mathbf{j}$

d) $\mathbf{F} = \left(3x^2y + 2xy^2\right)\mathbf{i} + \left(x^3 + 2x^2y + 3y^2\right)\mathbf{j}$.

4. Let S be the surface $z = 4 - x^2 - y^2, z \ge -3$ and let $\mathbf{F} = (2xyz + 3z)\mathbf{i} + x^2y\mathbf{j} + \cos(xyz)e^x\mathbf{k}$. Calculate

$$\int\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}.$$

Hint: Observe that ∂S is the boundary of another surface.

5. Let S be the union of the surfaces $z = x^2 + y^2 - 1$ with $z \le 0$ and $x^2 + y^2 + z^2 = 1, z \ge 0$. Let $x(t) = (\cos t, \sin t, 0)$, $0 \le t \le 2\pi$. Calculate $\int\int_S (\nabla \times \mathbf{F}) \cdot d\mathbf{S}$. for $\mathbf{F}$ an arbitrary C^1 vector field using Stokes' theorem. Do the same using Gauss's theorem (that is the divergence theorem).

6. Let V be the solid cylinder $x^2 + y^2 \le 1.|z| \le 1$. Describe the boundary of V. Orient the boundary using the outward normal and use Gauss's theorem to calculate $\int\int_{\partial V} \mathbf{F} \cdot d\mathbf{S}$ with $\mathbf{F} = x\mathbf{k} + y\mathbf{j} + z\mathbf{i}$.