Consider the iterated integral $\int_{-1}^{1}\int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}}\int_{-\sqrt{1-z^2-y^2}}^{\sqrt{1-z^2-y^2}} dxdydz$.

a) Describe the solid whose volume this triple integral calculates. (You must explain your answer to get full credit.)

b) Calculate the volume. Hint: You can use the formula

$$\int_{-a}^{a}\sqrt{a^2-x^2}dx = \frac{\pi a^2}{2}.$$

(You must calculate the iterated integral to get full credit. Just writing down a memorized formula for the volume is not sufficient.)

Solution: a) The outer double integral is over the domain $y^2+z^2 \leq 1$ in the yz plane. For each (y,z) in that disc the possible x values are $-\sqrt{1-z^2-y^2} \leq x \leq \sqrt{1-z^2-y^2}$. That is $x^2 \leq 1-y^2-z^2$. So the region is contained in $x^2+y^2+z^2 \leq 1$. If (x,y,z) satisfy $x^2+y^2+z^2 \leq 1$ then all three inequalities are satisfied: $-1 \leq z \leq 1, -\sqrt{1-z^2} \leq y \leq \sqrt{1-z^2}, \sqrt{1-z^2-y^2} \leq x \leq \sqrt{1-z^2-y^2}$.

So the region is the ball of radius 1.

b)

$$\int_{-1}^{1}\int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}}\int_{-\sqrt{1-z^2-y^2}}^{\sqrt{1-z^2-y^2}} dxdydz = 2\int_{-1}^{1}\int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}}\sqrt{1-z^2-y^2}dydz.$$

Take $a=\sqrt{1-z^2}$ in the hint and we have

$$2\int_{-1}^{1}\int_{-\sqrt{1-z^2}}^{\sqrt{1-z^2}}\sqrt{1-z^2-y^2}dydz = \int_{-1}^{1}(1-z^2)\pi dz = \pi\left(2-\frac{2}{3}\right) = \frac{4\pi}{3}.$$