30 Apr 2012

Schwarzschild

$$\alpha^2 dx_0^2 - \beta^2\left(\sum_{i=1}^{3} dx_i^2\right)$$

$$\sum_{i=1}^{n} \varepsilon_i e^{\varphi_i} dx_i^2$$

Want to calculate w_i^j where we can take $\theta_i = e^{\varphi_i} dx_i$

$$\begin{aligned} d\theta_i &= e^{\varphi_i} \sum_{j=1}^{n} \frac{\partial \varphi_i}{\partial x_j} dx_j \wedge dx_i \\ &= e^{\varphi_i} \sum_{j=1}^{n} \frac{\partial \varphi_i}{\partial x_j} e^{-\varphi_i} \theta_j \wedge dx_i \\ &= \sum_{j=1}^{n} \theta_j \wedge \left(e^{\varphi_i - \varphi_j} \frac{\partial \varphi_i}{\partial \varphi_j} dx_i\right) \end{aligned}$$

$$\rightsquigarrow \left.\begin{aligned} w_i^j &= e^{\varphi_i - \varphi_j} \frac{\partial \varphi_i}{\partial x_j} dx_i + \alpha\, dx_j \\ w_j^i &= e^{\varphi_j - \varphi_i} \frac{\partial \varphi_j}{\partial x_i} dx_j + \beta\, dx_i \end{aligned}\right\} \varepsilon_i w_i^j = -\varepsilon_j w_j^i$$

So $\varepsilon_j e^{\varphi_j - \varphi_i} \frac{\partial \varphi_i}{\partial x_j} dx_i + \varepsilon_i \alpha\, dx_j = -\varepsilon_i e^{\varphi_j - \varphi_i} \frac{\partial \varphi_j}{\partial x_i} dx_j - \varepsilon_i \beta\, dx_i$

$$\therefore \quad \alpha = -\varepsilon_i \varepsilon_j e^{\varphi_j - \varphi_i} \frac{\partial \varphi_j}{\partial x_i}$$

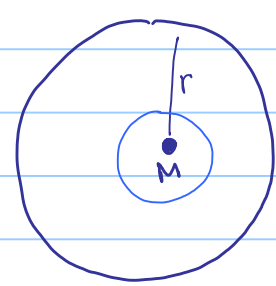
$$w_i^j = e^{\varphi_i - \varphi_j} \frac{\partial \varphi_i}{\partial x_j} dx_i - \varepsilon_i \varepsilon_j e^{\varphi_j - \varphi_i} \frac{\partial \varphi_j}{\partial x_i} dx_j$$

Schwarzschild metric:

Newton's gravitional constant.

mass of the object

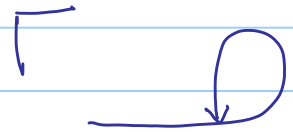
$$\frac{\left(1 - \frac{GM}{2c^2(x_1^2 + x_2^2 + x_3^2)^{1/2}}\right)^2}{\left(1 + \frac{GM}{2c^2(x_1^2 + x_2^2 + x_3^2)^{1/2}}\right)^2} dx_0^2 - \frac{1}{c^2}\left(1 + \frac{GM}{2c^2(x_1^2 + x_2^2 + x_3^2)^{1/2}}\right)^4 \left(dx_1^2 + dx_2^2 + dx_3^2\right)$$



A closed submfd is a closed subspace X equipped with a C^∞ structure. $\mathcal{O}_1$

$N=(X,\mathcal{O}_1)$ s.t. $i_N : N \to M$ is C^∞

$(di_N)_x : T_xN \to T_xM$ is surjective



In Lie theory it is necessary to allow submanifolds that are not topological subspaces such as the picture above and the irrational line in a torus.

Lemma M_1, M are C^∞ manifolds and $N \subseteq M$ is a closed submfd, $N_1 \subseteq M_1$ closed mfd.

$\psi : M_1 \to M$ C^∞ such that $\psi(N_1) \subseteq N$,

then $\psi|_{N_1}$ is C^∞ from N_1 to N.

Pf: Two step: (a) $\psi|_{N_1} : N_1 \to M$ is C^∞ Exercise!

(b) for $p \in N_1$, $q = \psi(p)$

$(V, \{x_1, \dots, x_n\})$ to be a coord. nbd. of q

such that $\exists$ coord. nbd $(W, \{z_1, \dots, z_s\})$ of N s.t. $\psi(V) \subseteq U$ $\left(\begin{smallmatrix}\text{in fact}\\ N\cap V\end{smallmatrix}\right)$

$di_N : T_q(N) \to T_q(M)$ is injective

So if $\dim N = k$, we may assume $dx_{1q} \circ (di_N)_q, \dots, dx_{kq} \circ (di_N)_q$ form a basis of T_qN.

By Inverse function theorem, $q \in U \subset V \subset M$ open s.t.

$y_1 \equiv x_1|_{U\cap N}, \dots, y_k \equiv x_k|_{U\cap N}$ form a system of coord.

$p \in \underbrace{\psi^{-1}(U) \cap V}_{= Z} \xrightarrow{\psi} U$

$x_i \circ \psi$ is C^∞ in Z.

$x_i \circ \psi = x_i \circ i_N \circ \psi = y_i \circ \psi \in C^\infty(V \cap N)$

4 May 2012

Lie algebras

G: Lie group

$L_g : G \to G$, $L_g x = gx$

$l_g : C^\infty(G) \to C^\infty(G)$, $l_g f = f \circ L_{g^{-1}}$

$X \in \mathrm{Vect}(G)$ is said to be left invariant if $(dL_g)_x X_x = X_{gx}$

$\Leftrightarrow X \circ l_g = l_g \circ X$

$$(X(l_g f))(x) = X_x(f \circ L_{g^{-1}}) = (dL_{g^{-1}})_x X_x \cdot f$$
$$(l_g(Xf))(x) = (Xf)(L_{g^{-1}}x) = X_{g^{-1}x} f$$

If X, Y are left invariant, so is $[X,Y] = XY - YX$

$$l_g^{-1}[X,Y]l_g = l_g^{-1}XYl_g - l_g^{-1}YXl_g = \underbrace{l_g^{-1}Xl_g}_{X}\underbrace{l_g^{-1}Yl_g}_{Y} - \underbrace{l_g^{-1}Yl_g}_{Y}\underbrace{l_g^{-1}Xl_g}_{X}$$
$$= [X,Y]$$

Define the Lie alg. of G to be the set of all left invariant vector fields. It's a real vector space. Denote it $\mathrm{Lie}(G)$.

Lemma The map $\mathrm{Lie}(G) \to T_eG$ given by $X \mapsto X_e$ is a linear isomorphism (i.e. $\dim \mathrm{Lie}(G) = \dim G$)

Pf: If $X_e = 0$, then $0 = (dL_g)_e X_e = X_g$

So the map is injective

If $V \in T_e(G)$, $\sigma : (-\varepsilon, \varepsilon) \to G$ is C^∞

$\sigma(0) = e$, $\sigma'(0) = V$

Define $X_g f = \frac{d}{dt} f(g\sigma(t))|_{t=0}$. $f \in C^\infty(G)$

If $\Phi(g,t) = g\sigma(t)$

then $\Phi : G \times (-\varepsilon, \varepsilon) \to G$ is C^∞

$X_g f = \frac{\partial}{\partial t} f(\Phi(g,t))|_{t=0}$

$X_e = V$

E.g. $G = GL(n,\mathbb{R})$ Describe $Lie(G)$.

$T_I(GL(n,\mathbb{R})) \sim M_n(\mathbb{R})$

If $A \in M_n(\mathbb{R})$, then

define a curve $\sigma(t)$ through I such that $\sigma(0) = I$, $\sigma'(0) = A$

e.g. $I + tA$ for $|t| < \varepsilon$ for some $\varepsilon > 0$.

Consider $\frac{d}{dt} f(g(I+tA))|_{t=0}$

for $g = \sum x_{ij}(g) E_{ij}$, $E_{ij} = [\delta_{ri}, \delta_{rs}]$

$A = \sum a_{rs} E_{rs}$

$$g(I+tA) = \sum x_{ij}(g) E_{ij} \left(I + t \sum a_{rs} E_{rs} \right)$$

$$= \sum_{i,j} x_{ij}(g) E_{ij} + t \sum_{\substack{i,j \\ r,s}} x_{ij}(g) a_{rs} \underbrace{E_{ij} E_{rs}}_{\delta_{jr} E_{is}}$$

$$= g + t \sum_{i,j,s} x_{ij} a_{js} E_{is}$$

$$\therefore \frac{d}{dt} f(g(I+tA))|_{t=0} = \sum a_{js} (D_{js} f)(g)$$

$\frac{d}{dt}|_{t=0} f(x + t e_i) = \frac{\partial}{\partial x_i} f(x)$

$$D_{js} = \sum x_{ij} \frac{\partial}{\partial x_{js}}$$

Exercise

$$[D_{ij}, D_{rs}] = D_{ij} D_{rs} - D_{rs} D_{ij} = \delta_{jr} D_{is} - \delta_{si} D_{rj}$$

$$[E_{ij}, E_{rs}] = \delta_{jr} E_{is} - \delta_{si} E_{rj}$$

$A \in T_I(GL(n,\mathbb{R})) = M_n(\mathbb{R}) \longrightarrow Lie(G)$

is a Lie alg. isomorphism. ∎

$GL(n,\mathbb{C}) \hookrightarrow GL(2n,\mathbb{R})$

In general, if G is a Lie group, we can put a Lie alg structure on $T_e(G)$

i.e. $[X_e, Y_e] = [X, Y]_e$

Suppose $H \subset G$ Lie group.

H is closed subgroup of G and closed submanifold.

We call this a closed Lie subgroup.

$i_H : H \longrightarrow G$ C^∞ and $(di_H)_h$ is injective.

If $X, Y \in Lie(H)$, $(di_H)_e(X_e) = \tilde{X}_e$, $\tilde{X}_e, \tilde{Y}_e \in Lie(G)$

$(di_H)_e(Y_e) = \tilde{Y}_e$

Assert $[\tilde{X}, \tilde{Y}]_e = (di_H)_e([X, Y]_e)$

Let $\sigma, \mu : (-\varepsilon, \varepsilon) \longrightarrow H$ be C^∞ s.t. $\sigma(0) = \mu(0) = e$, $\dot{\sigma}(0) = X_e$, $\dot{\mu}(0) = Y_e$

$$X_h f = \frac{d}{dt} f(h\sigma(t)) \qquad h \in H, f \in C^\infty(H)$$

$$\tilde{X}_g f = \frac{d}{dt}\Big|_{t=0} f(g\sigma(t)) \qquad g \in G, f \in C^\infty(G)$$

$$\tilde{X}_I(\tilde{Y} f) - \tilde{Y}_I(\tilde{X} f) = \frac{d}{dt}\Big|_{t=0} (\tilde{Y} f)(\sigma(t)) - \frac{d}{dt}\Big|_{t=0} (\tilde{X} f)(\mu(t))$$

$$= \frac{\partial}{\partial t}\Big|_{t=0} \frac{\partial}{\partial s}\Big|_{s=0} f(\sigma(t)\mu(s)) - \frac{\partial}{\partial t}\Big|_{t=0} \frac{\partial}{\partial s}\Big|_{s=0} f(\mu(t)\sigma(s))$$

may assume f is only defined in coordinate nbd $(U, \{x_1, \dots, x_n\})$ of e.

may assume $f = x_j \leftsquigarrow \left(\begin{array}{l} X = \sum a_i \frac{\partial}{\partial x_i} \\ X x_j = a_j \end{array} \right)$

Also shown we may assume $x_1|_{U \cap H}, \dots, x_m|_{U \cap H}$ form coord. on $U \cap H$

If $f = x_i$ for $i \leq m$ $(m = \dim H)$

$\Rightarrow [X, Y]_e = [\tilde{X}, \tilde{Y}]_e$ $\blacksquare$

$$I_{p,q} = \begin{pmatrix} I_p & 0 \\ 0 & -I_q \end{pmatrix} \qquad p + q = n$$

$$O(p,q) = \{ g \in M_n(\mathbb{R}) \mid g I_{p,q} g^T = I_{p,q} \} \qquad \dim = \frac{n(n-1)}{2}$$

$$U(p,q) = \{ g \in M_n(\mathbb{C}) \mid g I_{p,q} g^* = I_{p,q} \} \qquad \dim = n^2$$

$$O(p,q) \leftrightarrow Lie(G)_I = \{ X \in M_n(\mathbb{R}) \mid X I_{p,q} + I_{p,q} X^T = 0 \} = \mathfrak{so}(p,q)$$

$$U(p,q) \leftrightarrow Lie(G)_I = \{ X \in M_n(\mathbb{C}) \mid X I_{p,q} + I_{p,q} X^* = 0 \} = \mathfrak{u}(p,q)$$

Both spaces are closed under commutator

$$Lie(O(p,q)) \cong \mathfrak{so}(p,q)$$

$$Lie(U(p,q)) \cong \mathfrak{u}(p,q)$$

from Mon: for $N \subseteq M$ closed submanifold, $N \xrightarrow{\psi} M$ C^∞

$N_1 \subseteq N$ closed submfd s.t. $\psi(N_1) \subseteq N_1$

then $\psi : N_1 \to M$

Apply on Lie group:

Each of $O(p,q)$, $U(p,q)$ is a closed submfd. of $GL(n,\mathbb{R})$ or $GL(n,\mathbb{C})$

$$H \times H \to H$$

$$(h_1, h_2) \mapsto h_1 h_2 \qquad \text{NEED } C^\infty$$

$$H \to H$$

$$h \mapsto h^{-1}$$

Now $H \times H \xrightarrow{\text{mult.}} GL(n,\cdot)$ $\quad C^\infty$

$H \xrightarrow{\text{inverse}} GL(n,\cdot)$ $\quad C^\infty$

values in H. So it's C^∞ into H