Practice Problems for 104a Final Exam

1. Calculate the GCD of 3822 and 4185.

2. Find all integral solutions of the following equations in unknowns x, y.
a) $21x + 51y = 27$.
b) $12x + 20y = 5$.

3. Let $a, b \in \mathbb{Z}$ be such that $\gcd(a, b) = 1$. Show that if $ab = m^3$ with $m \in \mathbb{Z}$ them there exist $x, y \in \mathbb{Z}$ so that $a = x^3$ and $b = y^3$.

4. Prove that if $a, b \in \mathbb{Z}$ and $a^3 | b^2$ then $a | b$.

5. Show that if $N \in \mathbb{Z}$ and $N \geq 24$ then there exist $x, y \in \mathbb{N}$ so that $N = 5x + 7y$. Show that there are exactly 12 elements of $\mathbb{N}$ that cannot be written in the form $7x + 11y$ with $x, y \in \mathbb{N}$. (Hint: The following is a table of numbers $5x + 7y$ with the number placed in the xth row and the yth column with the labels starting with 0.

0	5	10	15	20	25	30
7	12	17	22	27	32	37
14	19	24	29	34	39	44
21	26	31	36	41	46	51
28	33	38	43	48	53	58

Observe that $24, 25, 26, 27, 28$ are in the grid.)

6. Suppose that you have only pennies (1 cent), dimes (10 cents) and quarters (25 cents) and you want to buy something that costs \$3.00. Prove that you can do this with 36 of your coins but not with 35.

7. For which primes p does $x^2 + x + 1 = 0$ have a solution in F_p? If it has solution in F_p how many does it have?

8. Find an integer that has a remainder of 9 when divided by 10 or 11 but is divisible by 13.

9. How many positive divisors does 14175 have? (Hint: It's largest prime divisor is 7.)

10. Consider the field F_{11} show that 2 is a primitive element (root) of F_{11}. Find all of the primitive elements of F_{11}.

11. Let p be an odd prime and let $\left(\frac{a}{p}\right)$ be the Legendre symbol. Prove that

$$\sum_{i=1}^{p-1} \left(\frac{i}{p}\right) = 0.$$

(Hint: Write this sum using a primitive element.)

12. Calculate the following Legendre symbols.
a) $\left(\frac{229}{379}\right)$ b) $\left(\frac{7919}{12553}\right)$.