Solutions to Practice Final

December 8, 2007

Problem 1.

P	Q	[(not P) or Q] $\Longrightarrow$ [P $\Longrightarrow$ Q]	[P or Q] and [not Q]	[P $\Longrightarrow$ Q] or [Q $\Longrightarrow$ P]
T	T	T	F	T
T	F	T	T	T
F	T	T	F	T
F	F	T	F	T

Since statements a) and c) are always true (independent of the truth values assigned to P and Q) they are tautologies.

Problem 2.

We use "strong" induction on n.

Base case: We must show that $u_{11} \geq (\frac{3}{2})^{11}$ and $u_{12} \geq (\frac{3}{2})^{12}$. Here are the first twelve terms of the Fibonacci sequence:

$$1, 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144$$

Since $(\frac{3}{2})^{11} = 86.49...$ we see that $u_{11} \geq (\frac{3}{2})^{11}$. Furthermore

$$(\frac{3}{2})^{12} < (87)(\frac{3}{2}) < 144$$

so we get that $u_{12} \geq (\frac{3}{2})^{12}$.

Inductive hypothesis: Assume that $u_{n-1} \geq (\frac{3}{2})^{n-1}$ and $u_{n-2} \geq (\frac{3}{2})^{n-2}$ for some $n \geq 13$.

Inductive step: We must show that $u_n \geq (\frac{3}{2})^n$. To wit

$$\begin{aligned}
u_n &= u_{n-1} + u_{n-2} \\
&\geq (\frac{3}{2})^{n-1} + (\frac{3}{2})^{n-2} \\
&= (\frac{3}{2})^{n-2}(\frac{3}{2} + 1) \\
&= (\frac{3}{2})^{n-2}(\frac{5}{2}) \\
&\geq (\frac{3}{2})^n
\end{aligned}$$

(In the last equality we use the simple fact that $\frac{5}{2} = \frac{10}{4} > \frac{9}{4} = (\frac{3}{2})^2$.)

By induction the proof is complete.

Problem 3.

+	0	1	2	3	4	5	6	7
0	0	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7	0
2	2	3	4	5	6	7	0	1
3	3	4	5	6	7	0	1	2
4	4	5	6	7	0	1	2	3
5	5	6	7	0	1	2	3	4
6	6	7	0	1	2	3	4	5
7	7	0	1	2	3	4	5	6

·	0	1	2	3	4	5	6	7
0	0	0	0	0	0	0	0	0
1	0	1	2	3	4	5	6	7
2	0	2	4	6	0	2	4	6
3	0	3	6	1	4	7	2	5
4	0	4	0	4	0	4	0	4
5	0	5	2	7	4	1	6	3
6	0	6	4	2	0	6	4	2
7	0	7	6	5	4	3	2	1

All solutions to $x^2 \equiv 1 (\bmod 8)$ are $x \equiv 1, 3, 5, 7 (\bmod 8)$. If x is invertible in $\mathbb{Z}_8$ then $x^{-1} \equiv x (\bmod 8)$.

Problem 4.

Suppose $x \equiv 1 (\bmod m)$. Then there exists q such that

$$x = qm + 1$$

Then

$$x^2 = q^2m^2 + 2qm + 1 = (q^2m + 2q)m + 1$$

so indeed $x^2 \equiv 1 (\bmod m)$. If $x \equiv m - 1 (\bmod m)$ then there exists q such that

$$x = qm + (m - 1)$$

Then

$$\begin{aligned} x^2 &= q^2m^2 + 2qm(m-1) + (m^2 - 2m + 1) \\ &= (q^2m + 2q(m-1) + m - 2)m + 1 \end{aligned}$$

so indeed $x^2 \equiv 1 (\bmod m)$. Now suppose m is prime and $x^2 \equiv 1 (\bmod m)$. We want to show that this implies that $x \equiv 1 (\bmod m)$ or $x \equiv m - 1 (\bmod m)$. We prove this by contradiction: assume $x \not\equiv 1, m - 1 (\bmod m)$. Certainly we cannot have $x \equiv 0 (\bmod m)$ since then $x^2 \equiv 0 (\bmod m)$ as well, contradicting our hypothesis. Then $x \equiv j (\bmod m)$ where $1 < j < m-1$. Since $x^2 \equiv 1 (\bmod m)$ this says that $j^2 \equiv 1 (\bmod m)$ so we can write

$$j^2 = qm + 1$$

or

$$(j+1)(j-1) = qm$$

Since m is prime this implies that m divides $j-1$ or $j+1$. But

$$1 \leq j-1, j+1 \leq m-1$$

so m cannot divide $j-1$ nor $j+1$, a contradiction. Therefore our assumption that $x \not\equiv 1, m-1 (\bmod m)$ must be false. Notice that this is not true for m not prime. As we saw in problem 3 above, 3^2 and 5^2 are both congruent to 1 modulo 8.

Problem 5.

In part a) when we plug in $n = 3$ we get $1 + \frac{1}{2} + \frac{1}{3} = \frac{11}{6} = \frac{22}{12}$ on the left hand side, while on the right hand side we have $\frac{6(3^2)-13(3)+11}{4} = \frac{26}{4} = \frac{78}{12}$. In part c) when we plug in $n = 2$ we get $(1 - \frac{1}{3})^2 = \frac{4}{9}$ which is stricly less than $\frac{1}{2}$. So a) and c) are both false. We prove b) by induction on n.

Base case:

$$1 + \frac{1}{2} = 2 - \frac{1}{2}$$

Inductive hypothesis: assume that

$$1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-1}} = 2 - \frac{1}{2^{n-1}}$$

Inductive step:

$$\begin{aligned} 1 + \frac{1}{2} + \cdots + \frac{1}{2^n} &= 1 + \frac{1}{2} + \cdots + \frac{1}{2^{n-1}} + \frac{1}{2^n} \\ &= (2 - \frac{1}{2^{n-1}}) + \frac{1}{2^n} \\ &= 2 - (\frac{1}{2^{n-1}} - \frac{1}{2^n}) \\ &= 2 - \frac{1}{2^n} \end{aligned}$$

By the induction the proof is complete.

Problem 6.

Clearly $n = 0$ is in the set. Suppose $n \neq 0$ and $36n^2 \geq n^4$. Then

$$36n^2 \geq n^4 \Longleftrightarrow 36 \geq n^2 \Longleftrightarrow 6 \geq |n|$$

So a nonzero integer is in the set if and only if its absolute value is less than or equal to 6. There are 12 such integers. Combining this with zero we see that our set has 13 elements.

Problem 7.

The function f is injective since

$$\frac{x}{1-x} = \frac{x'}{1-x'} \Longrightarrow x(1-x') = x'(1-x) \Longrightarrow x = x'$$

It is surjective since given $y \in \mathbb{R}^+$ let $x = \frac{y}{1+y}$. Then $0 < x < 1$ and $f(x) = y$.

Problem 8.

Step 1: Decide if a solution exists and if so how many.

$$\begin{aligned} 63 &= 54 \cdot 1 + 9 \\ 54 &= 9 \cdot 6 \end{aligned}$$

So $(63, 54) = 9$. Since 9 divides 36 a solution exists, and there are precisely 9 distinct solutions modulo 63.

Step 2: Find a particular solution.

$$\begin{aligned} 54(-1) + 63(1) &= 9 \\ 54(-4) + 63(4) &= 36 \end{aligned}$$

So $x_p \equiv -4 \equiv 59 \pmod{63}$ is a particular solution,

Step 3: Solve the homogenous equation.

$$\begin{aligned} 54x_h + 63q &= 0 \\ 6x_h + 7q &= 0 \end{aligned}$$

so $x_h = 7y$ and $q = -6y$ is the general solution to the homogenous equation (where y is any integer). As we saw above there are 9 distinct solutions modulo 63. We get them by plugging in $y = 0, ..., 8$ into the formula for x_h:

$$x_h = 0, 7, 14, 21, 28, 35, 42, 49, 56$$

Step 4: Write down the general solution. We just calculate $x_p + x_h \pmod{63}$ for all the different $x_h's$:

$$x \equiv 3, 10, 17, 24, 31, 38, 45, 52, 59 \pmod{63}$$

Problem 9. Consider the following calculations:

$$
\begin{aligned}
2^8 &\equiv 256 \equiv 33 (\bmod 223) \\
2^{16} &\equiv (2^8)^2 \equiv 33^2 \equiv 197 (\bmod 223) \\
197 &\equiv -26 (\bmod 223) \\
2^{32} &\equiv (-26)^2 \equiv 7 (\bmod 223) \\
2^{37} &\equiv 2^{32} 2^5 \equiv 7 \cdot 32 \equiv 1 (\bmod 223)
\end{aligned}
$$

So indeed $2^{37} \equiv 1 (\bmod 223)$ and so $2^{37} - 1$ divides 223.