Math 104A, Number Theory, Fall 2002.
Homework extra credit question 2.1 #25.

Suppose that we copy the Fibonacci construction but set

$$x_1 = a, \qquad x_2 = b.$$

Then

$$x_3 = a + b, \qquad x_4 = a + 2b, \qquad x_5 = 2a + 3b, \qquad \ldots \qquad x_k = f_{k-2}a + f_{k-1}b.$$

Applying this to the situation

$$a = f_{d-1}, \qquad b = f_d,$$

we get the equation

$$f_{d+k} = x_{k+2} = f_k f_{d-1} + f_{k+1} f_d.$$

In particular, we can see by induction that $f_d | f_{nd}$ for all n. Indeed, this holds for $n = 1$, and

$$f_{kd+d} = f_{kd} f_{d-1} + f_{kd-1} f_d,$$

so if $f_d | f_{kd}$ then $f_d | f_{(k+1)d}$, and by induction, $f_d | f_{nd}$ for every n.

We claim that the Fibonacci numbers which are divisible by f_d are precisely the numbers f_{nd}. To show that there are no others, we first show that if $(d, e) = 1$, then $(f_d, f_e) = 1$. Indeed, choose the smallest $d > 1$ such that there exists e with $(d, e) = 1$ and $(f_d, f_e) > 1$. Now with this d fixed, pick the smallest value of e with $(d, e) = 1$ and $(f_d, f_e) > 1$. Then $e > d$ or else we could have switched d with e to get a smaller value of d to start with. Write $e = qd + r$ with $0 < r < d$. Then

$$f_e \;=\; f_{qd+r} \;=\; f_{qd} f_{r-1} \;+\; f_{qd-1} f_r.$$

Now $f_d | f_{qd}$ so

$$(f_d, f_e) = (f_d, f_{qd-1} f_r).$$

However, $qd - 1 < e$ and $(qd - 1, d) = 1$, so $(f_{dq-1}, f_d) = 1$. Similarly, $(f_r, f_d) = 1$, and hence $(f_e, f_d) = 1$ which is a contradiction. Hence we must have $(f_d, f_e) = 1$ whenever $(d, e) = 1$.

Now we see that the only Fibonacci numbers divisible by f_d are the numbers f_{nd}. Indeed, suppose that $f_d | f_c$ where $c = qd + r$ with $0 < r < d$. Then

$$f_c \;=\; f_{qd+r} \;=\; f_{qd} f_{r-1} \;+\; f_{qd-1} f_r.$$

Since $f_d | f_c$ and $f_d | f_{qd}$ we get

$$f_d | f_{qd-1} f_r,$$

but $(d, qd - 1) = 1$, so $f_d | f_r$. However, since $r < c$, this is impossible. Hence we cannot find such a value c, which proves our assertion.