Math 104A, Number Theory, Fall 2002.
Summary of Lecture 3.

Binomial Coefficients.
Last time we gave a definition of the Binomial Coefficients. However, it was quite unpopular, so let's start again and give a different definition.

Josh's Definition.

$$\binom{n}{r} \;=\; \frac{n!}{r!(n-r)!}.$$

Proposition. *(a).* $\binom{n}{0} \;=\; \binom{n}{n} \;=\; 1.$

(b). $\binom{n}{r} \;=\; \binom{n-1}{r-1} \;+\; \binom{n-1}{r}.$

This can be illustrated with Pascal's triangle.

Proof. (a). This is straight forward if you know $0! = 1$.

(b).

$$\begin{aligned}\binom{n-1}{r-1} + \binom{n-1}{r} &= \frac{(n-1)!}{(r-1)!(n-r)!} + \frac{(n-1)!}{r!(n-r-1)!}\\ &= \frac{(n-1)!}{r!(n-r)!}(r+n-r) \;=\; \binom{n}{r}.\end{aligned}$$

Theorem.

$$(*) \qquad (a+b)^n \;=\; \binom{n}{0} b^n \;+\; \binom{n}{1} ab^{n-1} \;+\; \binom{n}{2} a^2b^{n-2} \;+\; \ldots \;+\; \binom{n}{n} a^n.$$

Proof. By induction on n. Define

$$S \;=\; \{n \in \mathbb{N} : (*) \text{ holds.}\}.$$

Step 1. We will show that $0 \in S$.
Step 2. We will show that if $n \in S$ then $n+1 \in S$.
Step 3. By induction we can conclude that $S = \mathbb{N}$.

Step 1. $\binom{0}{0} \;=\; 0!/(0!\,0!) \;=\; 1$, since $0! = 1$.

Step 2. Suppose that $n \in S$ so (*) holds for n. Now

$$\begin{aligned}
(a+b)^{n+1} &= (a+b)(a+b)^n \\
&= (a+b)\left(\binom{n}{0}b^n + \binom{n}{1}ab^{n-1} + \binom{n}{2}a^2b^{n-2} + \ldots + \binom{n}{n}a^n\right) \\
&= \left(\binom{n}{0}b^{n+1} + \binom{n}{1}ab^n + \binom{n}{2}a^2b^{n-1} + \ldots + \binom{n}{n}a^nb\right) \\
&\qquad + \left(\binom{n}{0}ab^n + \binom{n}{1}a^2b^{n-1} + \binom{n}{2}a^3b^{n-2} + \ldots + \binom{n}{n}a^{n+1}\right) \\
&= \binom{n}{0}b^{n+1} + \left(\binom{n}{0}+\binom{n}{1}\right)ab^n + \left(\binom{n}{1}+\binom{n}{2}\right)a^3b^{n-2} \\
&\qquad + \ldots + \left(\binom{n}{n-1}+\binom{n}{n}\right)a^nb + \binom{n}{n}a^{n+1} \\
&= \binom{n+1}{0}b^{n+1} + \binom{n+1}{1}ab^n + \binom{n+1}{2}a^2b^{n-1} + \ldots + \binom{n+1}{n+1}a^{n+1}.
\end{aligned}$$

We now see immediately that

$$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n} = 2^n,$$

because

$$(1+1)^n = \sum_{r=0}^{n}\binom{n}{r}1^r1^{n-r}.$$

Division. We recalled the definition of divisability, and proved Lemma 2.1.3 that for integers a, b, c, x, y,

(a). If $a|b$ and $x|y$ then $ax|by$.

(b). If $a|b$ and $b|c$ then $a|c$.

(c). If $a|b$ and $b \neq 0$ then $|a| \leq |b|$.

(d). If $a|b$ and $a|c$ then $a|bx+cy$.

We then stated the **Division Theorem**: Given two integers a and b with $b \neq 0$, there exists unique integers q and r such that

$$a = bq + r, \qquad\qquad 0 \leq r < |b|.$$

Examples.

1. $a = 13$, $b = 4$, $13 = 4 \cdot 3 + 1$.
2. $a = -13$, $b = 4$, $-13 = 4 \cdot (-4) + 3$.
3. $a = 13$, $b = -4$, $13 = -4 \cdot (-3) + 1$.
4. $a = -13$, $b = -4$, $-13 = -4 \cdot 4 + 3$.