Math 104A, Practice Final, Fall 2002.

All numbers are assumed to be integers unless otherwise stated.

1. The Fibonacci numbers f_k are defined by $f_0 = 0$, $f_1 = 1$ and $f_{k+1} = f_{k-1} + f_k$. Show by induction on d that for $d \geq 0$ and $k \geq 0$,

$$f_{k+d+1} = f_{k+1} f_{d+1} + f_k f_d.$$

2. Write down the general solution (x, y) of the equation $7x + 8y = 50$, and determine all solutions with x and y both positive.

3. If s and t are positive and relatively prime and r_k is the remainder when ks is divided by t, then

$$r_0 + \cdots + r_{t-1} = at^2 + bt + c.$$

What are the rational numbers a, b and c? Explain.

4. List the numbers a with $0 \leq a < 20$ such that there is at least one solution x to the equation

$$ax \equiv 14 \pmod{20}.$$

5. Decide which of the following statements are true for every integer x, and all positive integers c and m, and justify your answer.

(i) $cx \equiv 0 \pmod{cm} \Rightarrow x \equiv 0 \pmod{m}$.

(ii) $x \equiv 0 \pmod{m} \Rightarrow cx \equiv 0 \pmod{cm}$.

(iii) $cx \equiv 0 \pmod{cm} \Rightarrow x \equiv 0 \pmod{cm}$.

(iv) $x \equiv 0 \pmod{cm} \Rightarrow x \equiv 0 \pmod{m}$.

6. Determine the prime factorization of 192 and find the smallest exponent e such that $x^e \equiv 1 \pmod{192}$ for every value of x.

* NOTE: This problem turns out to involve rather a lot of calculation. Such a problem will not be on the real test!

7. Determine the number of solutions to the equation $x^{16} \equiv 1 \pmod{41}$, and the number of solutions to the equation $x^{16} \equiv -1 \pmod{41}$.

8. Show that there are infinitely many primes of the form $6k + 1$.