$$C_4 = -1 + \frac{1}{2 + \frac{1}{3 + \frac{1}{7+1}}} = -1 + \frac{1}{2 + \frac{8}{25}} = -1 + \frac{25}{58} = -\frac{33}{58}$$

3. Given positive integers a, b, c, and d, show that $[a, c] < [a, d]$ for $c > d$, but $[a, b, c] > [a, b, d]$.

Well, $1/c < 1/d$ so $a + 1/c < a + 1/d$, and hence $[a, c] < [a, d]$. If you think about it, it is true that $[a, b, c] = [a, [b, c]]$. By the previous proof (with b instead of a), we know that $[b, d] > [b, c]$, and again by the previous proof (with c and d replaced by $[b, d]$ and $[b, c]$ respectively), we have $[a, b, d] = [a, [b, d]] < [a, [b, c]] = [a, b, c]$, which is what we wanted to prove.

4. Let $a_1, a_2, \ldots, a_n, x$ be positive real numbers. Determine the values of n for which $[a_0, a_1, \ldots, a_n] > [a_0, a_1, \ldots, a_n + x]$ holds.

If you think hard about what we proved in the last problem, you'll see that the inequality above only holds for n odd. In problem 3 we proved the case $n = 1$ (with $a = a_0$, $c = a_1 + x$, $d = a_1$). Now assume that n is even, and that we know that $[a_0, a_1, \ldots, a_{n-2}] > [a_0, a_1, \ldots, a_{n-2} + x]$ holds for any choice of $a_0, \ldots, a_{n-2}$ (we use $n - 2$ since it is the even number just before n). We use the fact that $[a_0, a_1, \ldots, a_n] = [a_0, a_1, [a_2, \ldots, a_n]]$. Our induction hypothesis tells us that $[a_2, a_3, \ldots, a_n] > [a_2, a_3, \ldots, a_n + x]$ (this is why we carefully specified that our induction hypothesis to hold for any choice of a_i). Now we can apply problem 3 to the case where $a = a_0$, $b = a_1$, $c = [a_2, a_3, \ldots, a_n]$, and $d = [a_2, a_3, \ldots, a_n + x]$ to arrive at:

$$[a_0, a_1, \ldots, a_n] = [a_0, a_1, [a_2, \ldots, a_n]] > [a_0, a_1, [a_2, \ldots, a_n + x]] = [a_0, a_1, \ldots, a_n + x]$$

7. Show that the denominator q_k of the kth convergent of $[a_0, a_1, \ldots, a_n]$ satisfies $q_k \geq 2^{k/2}$ for $k \geq 2$, if $a_1 \geq 1$ for all $i \geq 1$.

We prove this by induction using the formula $q_k = a_k q_{k-1} + q_{k-2}$. The base $k = 2$ case has $q_2 = a_2 q_1 + q_0 = 1 \cdot a_1 + 1 \geq 2 = 2^{2/2}$ because $a_i \geq 1$ for all $i \geq 1$. Because our recursive relates q_k to the previous two q_i, we should also prove the case $k = 3$.

$$q_3 = a_3 q_2 + q_1 \geq 1 \cdot 2 + a_1 \geq 3 \geq 2^{3/2} \approx 2.82843$$

Now assume that $q_i \geq 2^{i/2}$ for all $2 \leq i \leq k$. Then we can see that

$$q_{k+1} = a_{k+1} q_k + q_{k-1} \geq 1 \cdot 2^{k/2} + 2^{(k-1)/2} = 2^{(k-1)/2}(2^{1/2} + 1) \geq 2^{(k-1)/2} \cdot 2 = 2^{(k+1)/2}$$

11. Let $a/b = [a_0, a_1, \ldots, a_n]$ and let $C_k = p_k/q_k$ be the kth convergent of a/b. Suppose all the $a_i > 0$, then show that p_k form an increasing sequence and

$$\frac{p_n}{p_{n-1}} = [a_n, \ldots, a_0]$$

and

$$\frac{q_n}{q_{n-1}} = [a_n, \ldots, a_1]$$

We first use the recursive formula $p_k = a_k p_{k-1} + p_{k-2}$. If we allow the possibility that a_k can be very small, then we could concoct examples where $p_k < p_{k-1}$ (try the last two p_k of $[1, 2, 3, 1/2]$), so the text must mean that all the $a_i \geq 1$. Then we not only have $p_i > 0$ for all i, but we also have $p_k = a_k p_{k-1} + p_{k-2} \geq p_{k-1} + p_{k-2} > p_{k-1}$, so the p_k form an increasing sequence.

We prove the formulas by induction on n. Since we will use the recursive formula for p_k, we will have to prove the first two cases of the formula above. For $n = 1$, the formula above is $(a_1a_0 + 1)/a_0 = [a_1, a_0]$, which we can see is true. For $n = 2$, we can compute

$$\frac{p_2}{p_1} = \frac{a_2(a_1a_0 + 1) + a_0}{a_1a_0 + 1} = a_2 + \frac{a_0}{a_1a_0 + 1} = a_2 + \frac{1}{a_1 + \frac{1}{a_0}}$$

Now we assume that $p_k/p_{k-1} = [a_k, \dots , a_0]$ is true for all $k < n$. Then we can use the recursive formula $p_n = a_np_{n-1} + p_{n-2}$ to get the following (by dividing by p_{n-1})

$$\frac{p_n}{p_{n-1}} = a_n + \frac{p_{n-2}}{p_{n-1}} = a_n + \frac{1}{\frac{p_{n-1}}{p_{n-2}}} = a_n + \frac{1}{[a_{n-1}, \dots , a_0]} = [a_n, \dots , a_0]$$

The proof of the formula for q_n/q_{n-1} follows similarly.

18. Let C_k be the kth convergent of $a/b = [a_0, a_1, \dots , a_n]$. Show that each convergent is nearer to a/b than the previous one.

Our goal here is to prove that $|C_k - a/b| < |C_{k-1} - a/b|$ for all k. We start by proving that $|C_{k+1} - C_k| < |C_{k+1} - C_{k-1}|$, where we use Corollary 11.2.6 and Lemma 11.2.7. We would like to prove that

$$|C_{k+1} - C_k| = \frac{1}{q_{k+1}q_k} \leq \frac{a_{k+1}}{q_{k+1}q_{k-1}} = |C_{k+1} - C_{k-1}|$$

If we clear the denominators, we get $q_{k-2} \leq a_kq_{k-1}$. Because $a_k \geq 1$ and q_k are increasing, we know that $q_{k-2} \leq q_{k-1} \leq a_kq_{k-1}$, so we have proven that $|C_k - C_{k-1}| \leq |C_k - C_{k-2}|$.

Now we apply Corollary 11.2.8 so that we can get rid of the absolute values. Let us assume that k is odd, so that we have the following inequalities $C_{k-1} < C_{k+1} \leq a/b \leq C_k$. Then we have

$$\left|C_k - \frac{a}{b}\right| = C_k - \frac{a}{b} = C_k - C_{k+1} + C_{k+1} - \frac{a}{b} < C_{k+1} - C_{k-1} + \frac{a}{b} - C_{k-1} = \frac{a}{b} - C_{k-1} = \left|C_{k-1} - \frac{a}{b}\right|$$

Here we used the facts that $C_k - C_{k+1} < C_{k+1} - C_{k-1}$, $C_{k+1} - \frac{a}{b} \leq 0$, and $\frac{a}{b} - C_{k-1} > 0$ (the last two imply that $C_{k+1} - \frac{a}{b} < \frac{a}{b} - C_{k-1}$).