104B Problem Set 6

Rino Sanchez

March 12, 2003

11.2, # 5ab

5. Solve the following Diophantine equations.

(a) $-718x + 123y = 1$: Here we compute the continued fraction expansion of $718/123$ to be $[5, 1, 5, 6, 1, 2]$ and calculate the last two convergents, $[5, 1, 5, 6, 1] = 251/43$ and $718/123$. Because of Corollary 11.2.6, we know that

$$\frac{718}{123} - \frac{251}{43} = -\frac{1}{123 \cdot 43}$$

If we clear the denominators, we get the integer equation $718 \cdot 43 - 123 \cdot 251 = 1$, so we have the solution $x = -43$ and $y = -251$ to $-718x + 123y = 1$.

(b) $417x - 172y = 1$: Again we compute the continued fraction expansion of $417/172$ to be $[2, 2, 2, 1, 4, 5]$ and calculate the last two convergents, $[2, 2, 2, 1, 4] = 80/33$ and $417/172$. As before, we get the integer equation $417 \cdot 33 - 172 \cdot 80 = 1$, so $x = 33$ and $y = 80$ solves the equation.

11.3, # 1, 4

1. Let $x = [a_0, a_1, \ldots, a_k, \ldots]$ be an infinite simple continued fraction. Show that

$$x = [a_0, a_1, \ldots, a_k, [a_{k+1}, \ldots]]$$

Let $C_i = [a_0, \ldots, a_i]$. By definition, $[a_0, a_1, \ldots, a_k, \ldots] = \lim_{i \longrightarrow \infty} C_i$. For $i > k$, we know that $[a_0, a_1, \ldots, a_k, \ldots, a_i] = [a_0, a_1, \ldots, a_k, [a_{k+1}, \ldots, a_i]]$. Now we take the limit as $i \longrightarrow \infty$

$$[a_0, a_1, \ldots, a_k, \ldots] = \lim_{i \longrightarrow \infty} [a_0, a_1, \ldots, a_k, \ldots, a_i] = \lim_{i \longrightarrow \infty} [a_0, a_1, \ldots, a_k, [a_{k+1}, \ldots, a_i]]$$

Now comes a slightly delicate point in the argument. We want to prove that

$$\lim_{i \longrightarrow \infty} [a_0, a_1, \ldots, a_k, [a_{k+1}, \ldots, a_i]] = [a_0, a_1, \ldots, a_k, [a_{k+1}, \ldots]]$$

which by definition is $[a_0, a_1, \ldots, a_k, \lim_{i \longrightarrow \infty} [a_{k+1}, \ldots, a_i]]$. This means that we want to bring the limit inside of the continued fraction, which is the basic idea of continuity (as in continuous functions, not continued fractions). The function $f(z) = [a_0, a_1, \ldots, a_k, z]$ is a continuous function at $z > 0$ because it is the composition $f_0 \circ f_1 \circ \cdots \circ f_k(z)$ of the functions $f_i(z) = a_i + 1/z$, which are continuous for all $z > 0$. Then continuity tells us that if f is continuous at z_0 and z_i is a sequence converging to z_0, then $\lim_{i \longrightarrow \infty} f(z_i) = f(z_0)$. Now we simply let $z_0 = [a_{k+1}, \ldots]$ and $z_i = [a_{k+1}, \ldots, a_i]$ for $i > k$, and continuity finishes the argument:

$$\lim_{i \longrightarrow \infty} [a_0, a_1, \ldots, a_k, [a_{k+1}, \ldots, a_i]] = \lim_{i \longrightarrow \infty} f(z_i) = f(z_0) = [a_0, a_1, \ldots, a_k, [a_{k+1}, \ldots]]$$

4. If p/q is a convergent of the simple continued fraction expansion of an irrational number α, show that

$$\left|\alpha - \frac{p}{q}\right| < \frac{1}{q^2}$$

Let $\alpha = [a_0, \ldots]$ be the continued fraction expansion of α, $C_i = p_i/q_i$ the i-th convergent of α, and $p/q = p_k/q_k = C_k = [a_0, \ldots, a_k]$ be the k-th convergent. We can use Corollary 11.2.6 to write

$$\begin{aligned} \tfrac{p}{q} &= C_k = C_0 + (C_1 - C_0) + \cdots + (C_k - C_{k-1}) \\ &= C_0 + \tfrac{1}{q_1 q_0} - \tfrac{1}{q_2 q_1} + \cdots - \cdots + \tfrac{(-1)^{k-1}}{q_k q_{k-1}} = C_0 + \textstyle\sum_{i=1}^{k} \tfrac{(-1)^{i-1}}{q_i q_{i-1}} \end{aligned}$$

Similarly, $\alpha = C_0 + \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{q_i q_{i-1}}$, so that

$$\begin{aligned} \left|\alpha - \tfrac{p}{q}\right| &= \left|\textstyle\sum_{i=k+1}^{\infty} \tfrac{(-1)^{i-1}}{q_i q_{i-1}}\right| \\ &= \left|(-1)^k \left(\tfrac{1}{q_{k+1} q_k} - \tfrac{1}{q_{k+2} q_{k+1}} + \cdots\right)\right| \\ &= \left|\tfrac{1}{q_{k+1} q_k} - \tfrac{1}{q_{k+2} q_{k+1}} + \cdots\right| \end{aligned}$$

Now notice that because the q_k are an increasing sequence by Lemma 11.3.1,

$$\frac{1}{q_{i+1} q_i} - \frac{1}{q_{i+2} q_{i+1}} > 0$$

for all i, so the sum above is positive and we can get rid of the absolute value

$$\left|\alpha - \frac{p}{q}\right| = \frac{1}{q_{k+1} q_k} - \frac{1}{q_{k+2} q_{k+1}} + \cdots$$

Similarly, $-\frac{1}{q_{i+1} q_i} + \frac{1}{q_{i+2} q_{i+1}} < 0$, which implies that $-\frac{1}{q_{k+2} q_{k+1}} + \frac{1}{q_{k+3} q_{k+2}} + \cdots < 0$. Finally, because $q_{k+1} > q_k$, this gives us

$$\left|\alpha - \frac{p}{q}\right| = \frac{1}{q_{k+1} q_k} - \frac{1}{q_{k+2} q_{k+1}} + \cdots < \frac{1}{q_{k+1} q_k} < \frac{1}{q_k^2} = \frac{1}{q^2}$$

11.4, # 2, 3

2. Determine the continued fraction expansion of $\sqrt{17}$, $\sqrt{19}$, and $\sqrt{21}$.

We simply crunch the numbers from the formulas given in Theorem 11.4.1, which gives a fairly easy way to compute the continued fraction expansion of quadratic irrationals. Here are the final results, which all eventually repeat, giving $\sqrt{17} = [4, \overline{8}]$, $\sqrt{19} = [4, \overline{2, 1, 3, 1, 2, 8}]$, and $\sqrt{21} = [4, \overline{1, 1, 2, 1, 1, 8}]$.

k	A_k	B_k	x_k	a_k
0	0	1	$\sqrt{17}$	4
1	4	1	$4 + \sqrt{17}$	8
2	4	1	$4 + \sqrt{17}$	8

k	A_k	B_k	x_k	a_k
0	0	1	$\sqrt{19}$	4
1	4	3	$\frac{4+\sqrt{19}}{3}$	2
2	2	5	$\frac{2+\sqrt{19}}{5}$	1
3	3	2	$\frac{3+\sqrt{19}}{2}$	3
4	3	5	$\frac{3+\sqrt{19}}{5}$	1
5	2	3	$\frac{2+\sqrt{19}}{3}$	2
6	4	1	$4+\sqrt{19}$	8
7	4	3	$\frac{4+\sqrt{19}}{3}$	2

k	A_k	B_k	x_k	a_k
0	0	1	$\sqrt{21}$	4
1	4	5	$\frac{4+\sqrt{21}}{5}$	1
2	1	4	$\frac{1+\sqrt{21}}{4}$	1
3	3	3	$\frac{3+\sqrt{21}}{3}$	2
4	3	4	$\frac{3+\sqrt{21}}{4}$	1
5	1	5	$\frac{1+\sqrt{21}}{5}$	1
6	4	1	$4+\sqrt{21}$	8
7	4	5	$\frac{4+\sqrt{21}}{5}$	1

3. Determine the continued fraction expansions of $1+\sqrt{5}$ and $(2+\sqrt{7})/4$.
We repeat the same algorithm, and we find that $1+\sqrt{5} = [3, \overline{4}]$ and $(2+\sqrt{7})/4 = [\overline{1, 6, 5, 6}]$.

k	A_k	B_k	x_k	a_k
0	1	1	$1+\sqrt{5}$	3
1	2	1	$2+\sqrt{5}$	4
2	2	1	$2+\sqrt{5}$	4

k	A_k	B_k	x_k	a_k
0	2	4	$\frac{2+\sqrt{7}}{4}$	1
1	2	$\frac{3}{4}$	$\frac{8+4\sqrt{7}}{3}$	6
2	$\frac{5}{2}$	1	$\frac{5}{2}+\sqrt{7}$	5
3	$\frac{5}{2}$	$\frac{3}{4}$	$\frac{10+4\sqrt{7}}{3}$	6
4	2	4	$\frac{2+\sqrt{7}}{4}$	1