Math 104B, Number Theory, Winter 2003.

Lecture 15. Continued Fractions.

Continued Fractions.

$$a_0 \;+\; \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_{n-1} + \frac{1}{a_n}}}} \;=\; [a_0, a_1, \ldots, a_n].$$

This continued fraction is simple if $a_1, \ldots, a_n$ are greater than or equal to 1, and all are integers (sometimes we don't require a_n to be an integer).

Proposition 11.1.4. A real number x can be expressed as a finite simple continued fraction if and only if x is rational. (This is unique if we require $a_n > 1$ if $n > 1$.)
For a rational number x, the continued fraction expansion $x = [a_0, a_1, \ldots, a_n]$ is obtained as follows:

$$\begin{array}{rcl} & & a_0 = \lfloor x \rfloor, \\ x_1 = \dfrac{1}{\{x\}} & & a_1 = \lfloor x_1 \rfloor, \\ x_2 = \dfrac{1}{\{x_1\}} & & a_2 = \lfloor x_2 \rfloor \\ x_3 = \dfrac{1}{\{x_2\}} & & a_3 = \lfloor x_3 \rfloor \\ \vdots & & \vdots \end{array}$$

Here $\{x\} = x - \lfloor x \rfloor$ is the fractional part of x. The algorithm stops when $a_n = x_n$is an integer. We can obtain the algorithm by assuming that $x = [a_0, \ldots, a_n]$ and seeing what conditions $a_0, \ldots, a_n$ satisfy. First we note that if $n = 0$ then x is an integer,

$$x \;=\; \lfloor x \rfloor \;=\; a_0.$$

We can assume $n > 0$. Notice that

$$[a_0, \ldots, a_n] \;=\; a_0 + \frac{1}{[a_1, \ldots, a_n]}$$

If $n \geq 1$ then $1 \leq [a_1, \ldots, a_n] < \infty$ and so

(*) $$0 < \frac{1}{[a_1, a_2, \ldots, a_n]} \leq 1.$$

Since

$$[a_1, \ldots, a_n] \ = \ a_1 + \frac{1}{[a_2, \ldots, a_n]},$$

we see that (*) equals 1 if and only if $n = 1$ and $a_1 = 1$. Otherwise

$$0 < \frac{1}{[a_1, a_2, \ldots, a_n]} < 1$$

and $a_0 = \lfloor x \rfloor$, and setting $x_1 = 1/\{x\}$ we have

$$x \ = \ a_0 \ + \ \frac{1}{x_1}.$$

We can then continue to expand x_1 as a continued fraction. The case we have not considered is when $n = 1$ and $a_1 = 1$. In this case x is an integer and

$$x = [a_0, 1] = a_0 + \frac{1}{1}.$$

is a continued fraction expansion of x which will not arise if we just follow the algorithm. We can make the continued fraction expansion unique by placing a condition on the last term:

$$a_n > 1 \qquad \text{if} \qquad n > 0.$$

This now gives the expansion the algorithm will produce.

By considering the Euclidean algorithm for p, q where $x = p/q$, we see that the algorithm to expand rational $x = p/q$ as a continued fraction eventually terminates.

$$\begin{array}{llll}
p = a_0 q + r_1 & x = p/q, & \lfloor x \rfloor = a_0, & \{x\} = r_1/q \\
q = a_1 r_1 + r_2 & x_1 = q/r_1, & \lfloor x_1 \rfloor = a_1, & \{x_1\} = r_2/r_1 \\
r_1 = a_2 r_2 + r_3 & x_2 = r_1/r_2, & \lfloor x_2 \rfloor = a_2, & \{x_2\} = r_3/r_2 \\
\vdots & \vdots & & \vdots \\
r_{n-1} = a_n r_n + 0 & x_n = r_{n-1}/r_n, & \lfloor x_n \rfloor = a_n, & \{x_n\} = 0
\end{array}$$

Definition. For the continued fraction $[a_0, \ldots, a_n]$, we define $C_k = [a_0, \ldots, a_k]$.

Proposition 11.2.3. The numerator p_k and denominator q_k of C_k satisfy

(1) $$p_0 = a_0, \qquad p_1 = a_0 a_1 + 1$$

(2) $$q_0 = 1, \qquad q_1 = a_1,$$

(3) $$p_k = a_k p_{k-1} + p_{k-2}$$

(4) $$q_k = a_k q_{k-1} + q_{k-2}$$

(5) $$p_k q_{k-1} - q_k p_{k-1} = (-1)^{k-1}$$

$$p_k q_{k-2} - q_k p_{k-2} = (-1)^k a_k$$

Example. These recurrence formulas give us an improved method for solving linear diophantine equations. Indeed, if we want to solve $55x - 19y = 1$, we first carry out the Euclidean algorithm,

$$\begin{aligned} 55 &= 2 \cdot 19 + 17 \\ 19 &= 1 \cdot 17 + 2 \\ 17 &= 8 \cdot 2 + 1 \\ 2 &= 2 \cdot 1 + 0. \end{aligned}$$

So $55/19 = [2, 1, 8, 2]$. Then we use formulas (1), (2), (3), (4) to calculate the convergents

$$\frac{2}{1}, \qquad \frac{3}{1} \qquad \frac{8 \cdot 3 + 2}{8 \cdot 1 + 1} = \frac{26}{9}, \qquad \frac{2 \cdot 26 + 3}{2 \cdot 9 + 1} = \frac{55}{19}.$$

We use (5) to solve the equation and get

$$55 \cdot 9 - 19 \cdot 26 = (-1)^{3-1} = 1.$$