Math 104B, Number Theory, Winter 2003.

Lecture 17. Continued Fractions Continued.

Continued Fractions.

$$a_0 + \cfrac{1}{a_1 + \cfrac{1}{\ddots + \cfrac{1}{a_{n-1} + \cfrac{1}{a_n}}}} = [a_0, a_1, \ldots, a_n].$$

This continued fraction is simple if $a_1, \ldots, a_n$ are greater than or equal to 1, and all are integers (sometimes we don't require a_n to be an integer).

Last time:
Definition. For the continued fraction $[a_0, \ldots, a_n]$, we define $C_k = [a_0, \ldots, a_k]$.

Proposition 11.2.3. The numerator p_k and denominator q_k of C_k satisfy

(1) $$p_0 = a_0, \qquad p_1 = a_0 a_1 + 1$$

(2) $$q_0 = 1, \qquad q_1 = a_1,$$

(3) $$p_k = a_k p_{k-1} + p_{k-2}$$

(4) $$q_k = a_k q_{k-1} + q_{k-2}$$

(5) $$p_k q_{k-1} - q_k p_{k-1} = (-1)^{k-1}$$

$$p_k q_{k-2} - q_k p_{k-2} = (-1)^k a_k$$

Proof is by induction. Assume true for $k < n$.

$$\begin{aligned}[a_0, \ldots, a_n] = [a_0, \ldots, a_{n-1} + 1/a_n] &= \frac{(a_{k-1} + 1/a_k)p_{k-2} + p_{k-3}}{(a_{k-1} + 1/a_k)p_{k-2} + p_{k-3}} \\ &= \frac{a_k a_{k-1} p_{k-2} + a_k p_{k-3} + p_{k-2}}{a_k a_{k-1} q_{k-2} + a_k q_{k-3} + q_{k-2}} = \frac{a_k p_{k-1} + p_{k-2}}{a_k q_{k-2} + q_{k-2}}.\end{aligned}$$

For the proof of (5) we follow the matrix argument in the book, and (6) follows easily from (3), (4) and (6).

Corollary 11.2.6.

$$\begin{aligned} C_k - C_{k-1} &= \frac{(-1)^{k-1}}{q_k q_{k-1}} \\ C_k - C_{k-2} &= \frac{a_k (-1)^k}{q_k q_{k-2}}. \end{aligned}$$

Corollary 11.2.8. The odd convergents form a decreasing sequence and the even convergents form an increasing sequence. Each convergent lies between the two preceding ones.

Proposition 11.3.2. Let $a_0, a_1, \ldots, a_k, \ldots$ be a sequence of integers with $a_i \geq 1$ for $i \geq 1$. Let $C_k = [a_0, a_1, \ldots, a_k]$. Then $\lim_{k\to\infty} C_k$ exists.

Proposition 11.3.4. (a). If x is a real non-rational number, set $x_0 = x$ and recursively define

$$(*) \qquad a_n = \lfloor x_n \rfloor, \qquad x_{n+1} = \frac{1}{\{x_n\}}.$$

Then $x = [a_0, a_1, \ldots]$.

We did not prove this, instead we did an example.
Compute the continued fraction expansion of $\sqrt{11}$.
Solution. $3^2 < 11 < 4^2$ so $\lfloor \sqrt{11} \rfloor = 3$.

$$\begin{aligned}
x_0 &= \sqrt{11},\\
a_0 &= \left\lfloor \sqrt{11} \right\rfloor = 3,\\
x_1 &= \frac{1}{x_0 - a_0} = \frac{1}{\sqrt{11}-3} = \frac{\sqrt{11}+3}{2},\\
a_1 &= \left\lfloor \frac{\sqrt{11}+3}{2} \right\rfloor = \left\lfloor \frac{\lfloor\sqrt{11}\rfloor+3}{2} \right\rfloor = \left\lfloor \frac{6}{2} \right\rfloor = 3,\\
x_2 &= \frac{1}{x_1 - a_1} = \frac{1}{\frac{\sqrt{11}+3}{2} - 3} = \frac{2}{\sqrt{11}-3} = \sqrt{11}+3,\\
a_3 &= \left\lfloor \sqrt{11}+3 \right\rfloor = 6\\
x_3 &= \frac{1}{x_3 - a_3} = \frac{1}{\sqrt{11}+3-3} = \frac{1}{\sqrt{11}-3} = x_1.
\end{aligned}$$

We see that $\sqrt{11} = [3, 3, 6, 3, 6, 3, 6, 3, \ldots]$.