Math 104B, Number Theory, Winter 2003.

Lecture 17. Continued Fractions Continued.

Proposition 11.3.4. (a). If x is a real non-rational number, set $x_0 = x$ and recursively define

$$(*) \qquad a_n = \lfloor x_n \rfloor , \qquad x_{n+1} = \frac{1}{\{x_n\}}.$$

Then $x = [a_0, a_1, \ldots]$.

Example. Compute the continued fraction expansion of $\sqrt{7}$.
Solution. $2^2 < 7 < 3^2$.

$$\begin{aligned}
x_0 &= \sqrt{7}, \\
a_0 &= \left\lfloor \sqrt{7} \right\rfloor \; = \; 2, \\
x_1 &= \frac{1}{x_0 - a_0} \; = \; \frac{1}{\sqrt{7}-2} = \frac{\sqrt{7}+2}{3}, \\
a_1 &= \left\lfloor \frac{\sqrt{7}+2}{3} \right\rfloor \; = \; \left\lfloor \frac{\lfloor\sqrt{7}\rfloor+2}{3} \right\rfloor \; = \; \left\lfloor \frac{4}{3} \right\rfloor \; = \; 1, \\
x_2 &= \frac{1}{x_1 - a_1} \; = \; \frac{1}{\frac{\sqrt{7}+2}{3} - 1} \; = \; \frac{3}{\sqrt{7}-1} \; = \; \frac{\sqrt{7}+1}{2}, \\
a_2 &= \left\lfloor \frac{\sqrt{7}+1}{2} \right\rfloor \; = \; \left\lfloor \frac{\lfloor\sqrt{7}\rfloor+1}{2} \right\rfloor \; = \; \left\lfloor \frac{3}{2} \right\rfloor \; = \; 1, \\
x_3 &= \frac{1}{x_1 - a_1} \; = \; \frac{1}{\frac{\sqrt{7}+1}{2} - 1} \; = \; \frac{2}{\sqrt{7}-1} \; = \; \frac{\sqrt{7}+1}{3}, \\
a_3 &= \left\lfloor \frac{\sqrt{7}+1}{3} \right\rfloor \; = \; \left\lfloor \frac{\lfloor\sqrt{7}\rfloor+1}{3} \right\rfloor \; = \; \left\lfloor \frac{3}{3} \right\rfloor \; = \; 1, \\
x_4 &= \frac{1}{x_1 - a_1} \; = \; \frac{1}{\frac{\sqrt{7}+1}{3} - 1} \; = \; \frac{3}{\sqrt{7}-2} \; = \; \sqrt{7}+2, \\
a_4 &= \left\lfloor \sqrt{7}+2 \right\rfloor \; = \; 4, \\
x_1 &= \frac{1}{x_0 - a_0} \; = \; \frac{1}{\sqrt{7}-2} = x_1.
\end{aligned}$$

Hence $\sqrt{11} = [2, \overline{1, 1, 1, 4}]$. Now it is a fact that if $a_0 = \lfloor \sqrt{n} \rfloor$, then $\sqrt{n} + a_0$ is a purely periodic partial fraction. In our case, $\sqrt{11} + 2 = [4, \overline{1, 1, 1, 4}] = [\overline{4, 1, 1, 1}]$. We can also write this as $[4,1,1,1,\sqrt{11}+2]$, and we can use this idea to check the value of $[\overline{4,1,1,1}]$. We have that if $x = [\overline{4,1,1,1}]$ then $x = [4,1,1,1,x]$.

The partial fraction $x = [4,1,1,1,x]$ has convergents

$$\frac{4}{1}, \frac{5}{1}, \frac{9}{2}, \frac{14}{3}, \frac{14x+9}{3x+2} = x.$$

But then

$$3x^2 + 2x = 14x + 9 \Rightarrow 3x^2 - 12x - 9 = 0 \Rightarrow x^2 - 4x - 3 = 0 \Rightarrow (x-2)^2 = 7.$$

Hence $x = \pm\sqrt{7}+2$. But $[\overline{4,1,1,1}]$ is positive, so $x = \sqrt{7}+2$ as we computed.

Proof of Proposition 11.3.4(a). We will use two things we showed last time. Firstly for the continued fraction $[a_0, a_1, \ldots]$ with $a_j \geq 1$ for $j \geq 1$, the kth convergent $C_k = [a_0, \ldots, a_k] = p_k/q_k$ where $q_k \to \infty$ as $k \to \infty$ and p_k and q_k satisfy recurrence relations...., and

$$C_k - c_{k-1} \quad = \quad \frac{(-1)^{k-1}}{q_k q_{k-1}}.$$

(a). We have

$$x = [a_0, a_1, \ldots, a_{k-1}, x_k]. \tag{*}$$

This is easily seen by induction on k. Indeed, for $k = 0$, $x = [x_0]$ and if (*) holds for k then $x = [a_0, \ldots, a_{k-1}, x_k] = [a_0, \ldots, a_{k-1}, a_k + 1/x_{k+1}] = [a_0, \ldots, a_k, x_{k+1}]$ so (*) holds for $k+1$.
Now with $C_k = [a_0, \ldots, a_k]$, since the kth convergent of $[a_0, \ldots, a_k, x_{k+1}]$ is x, we have

$$|x - C_k| \quad = \quad \frac{1}{q_k(x_{k+1}q_k + q_{k-1})} \quad \leq \quad q_k \quad \to \quad 0, \qquad \text{as } k \to \infty.$$