Math 104B, Number Theory, Winter 2003.

Lecture 19. Continued Fractions Continued.

Matrix Techniques. Continued fractions can give information about for example sums of squares. Indeed if p_k/q_k is the kth convergent of $[a_0, \ldots, a_n, a_n, \ldots, a_0]$ then $p_{2n+1}^2 = p_n^2 + p_{n-1}^2$. This relation is seen most easily using matrix techniques.
Starting from scratch, note that

$$[a, s/t] = a + \frac{t}{s} \;=\; \frac{as+t}{s} \;=\; \frac{s'}{t'},$$

where

$$\begin{pmatrix} s' \\ t' \end{pmatrix} = \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix}.$$

Repeating this we get

$$[a_0, a_1, s/t] \;=\; \frac{s'}{t'}$$

where

$$\begin{pmatrix} s' \\ t' \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix}.$$

But

$$\begin{pmatrix} a \\ 1 \end{pmatrix} = \begin{pmatrix} a & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

so iterating, we obtain $[a_0, \ldots, a_n] = p_n/q_n$ where

$$\begin{pmatrix} p_n \\ q_n \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

We also see that

$$\begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_{n-1} & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} p_{n-1} \\ q_{n-1} \end{pmatrix},$$

and so

$$\begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix}.$$

From this we immediately obtain the recursion formulas for continued fractions.

$$\begin{pmatrix} p_{n-1} & p_{n-2} \\ q_{n-1} & q_{n-2} \end{pmatrix} \begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix}.$$

Taking the transpose of this relation, we also obtain

$$\begin{pmatrix} a_n & 1 \\ 1 & 0 \end{pmatrix} \cdots \begin{pmatrix} a_0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} p_n & q_n \\ p_{n-1} & q_{n-1} \end{pmatrix}$$

and so

$$\frac{p_n}{p_{n-1}} = [a_n, \ldots, a_0].$$

We also see that if the kth convergent of $[a_0, \ldots, a_n]$ is p_k/q_k and if $[b_0, \ldots, b_m] = p/q$, then

$$[a_0, \ldots, a_n, b_0, \ldots, b_m] = [a_0, \ldots, a_n, p/q] = \frac{p'}{q'}$$

where

$$\begin{pmatrix} p' \\ q' \end{pmatrix} = \begin{pmatrix} p_n & p_{n-1} \\ q_n & q_{n-1} \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}.$$

This can be applied to 11.2, #13.

Infinite Continued Fractions. We showed that if if $a_j \geq 1$ for every $j \geq 1$ then $[a_0, a_1, \ldots]$ converges. We also showed that if x is an irrational and we set $x_0 = x$, and recursively define $x_{k+1} = 1/\{x_k\}$, then set $a_k = \lfloor x_k \rfloor$, we have $x = [a_0, a_1, \ldots]$. Previously we showed that if x is rational then this procedure terminates, that is x_n is an integer for some n, and $x = [a_0, \ldots, a_n]$.

Proposition. If $x = [a_0, a_1, \ldots]$ where a_j are integers with $a_j \geq 1$ for every $j \geq 1$, then $a_k = \lfloor x_k \rfloor$ where x_k is defined recursively by $x_0 = x$ and $x_{k+1} = 1/\{x_k\}$.

This implies in particular that an infinite continued fraction is irrational, since for rational numbers the process terminates.

Proof of the proposition. We have

$$x = [a_0, a_1, \ldots] = [a_0, [a_1, \ldots]].$$

Now $[a_1, \ldots]$ is finite and since the even convergents form an increasing sequence, we have $1 < [a_1, \ldots] < \infty$ and so

$$a_0 < x = a_0 + \frac{1}{[a_1, \ldots]} < a_0 + 1.$$

Since a_0 is an integer, we see that $a_0 = \lfloor x \rfloor$. But then setting $x_1 = 1/\{x\}$, we have

$$x_1 = \frac{1}{x - a_0} = [a_1, \ldots,]$$

applying the previous result to x_1, we get $a_1 = \lfloor x_1 \rfloor$ and setting $x_2 = 1/\{x_1\}$ we have

$$x_2 = [a_2, \dots].$$

We can keep going to get the result. (It is easy to set up a formal induction.)

Notation. x_k is the kth complete quotient of $[a_0, a_1, \dots]$ and a_k is the kth partial quotient.

We have found the continued fraction expansion of $\sqrt{n}$ for several values of n. We will see that the irrational roots of quadratic polynomials are precisely the continued fractions which are eventually periodic. Before we study this, we look at the method of finding the continued fraction expansion of a root of other polynomials, specifically of polynomials which have a unique root.

Example. Find the first two terms in the continued fraction expansion of $5^{1/3}$.

Solution. $f(x) = x^3 - 5$. Then set $x_0 = 5^{1/3}$. This is the unique root of f.

Step 1 is to find a_0. Now since x_0 is the unique root of f and $f(x) \to \infty$ as $x \to \infty$, we have $f > 0$ for $x > x_0$ and $x < 0$ for $x < x_0$. Hence since $a_0 < x_0 < a_0 + 1$, we have $f(a_0) < 0$ and $f(a_0 + 1) > 0$. We compute $f(1) = -4$ and $f(2) = 3$, so $a_0 = 1$.

Step 2. As usual we set $x_1 = 1/\{x_0\} = 1/(x_0 - 1)$.

Step 3. Our aim is to transform the polynomial f into a polynomial f_1 whose unique root is x_1. To so this we first transform to a polynomial g whose unique root is $\{x_0\} = x_0 - a_0$. We do this easily by setting

$$g(x) = f(x + a_0).$$

In our case we have

$$g(x) = (x+1)^3 - 5 \;=\; x^3 + 3x^2 + 3x - 4.$$

Step 4. Is to transform g into the polynomial f_1 which has unique root $x_1 = 1/\{x_0\}$ and positive leading coefficient. This is done as follows: suppose that $g(x) = c_n x^n + \cdots + c_0$ where the leading coefficient $c_n > 0$. Then set

$$f_1(x) \;=\; -x^n g(1/x) \;=\; -(c_n + c_{n-1}x + \cdots + c_0 c^n). \tag{*}$$

In our case,

$$f_1(x) \;=\; -(1 + 2x + 3x^2 - 4x^3) \;=\; 4x^3 - 3x^2 - 3x - 1.$$

Repeat. We iterate this process. In particular we now find $a_1 = \lfloor x_1 \rfloor$. We know that $a_1 \geq 1$ which we can check by calculating $f_1(1) = -3$. Now $f_1(2) = 32-12-6-1 = 13$, so $a_1 = 1$.

Solution. $5^{1/3} = [1, 1, ?, \ldots]$. The process can be iterated to find more terms.

The only point we should check is that if the leading order coefficient of f is positive, then f_1 given by (*) has unique root $x_1 = 1/\{x_0\}$ and positive leading coefficient. It is not hard to see that the unique root is x_1. To show that the leading order coefficient is positive, note that the leading order coefficient of g is the same as the leading order coefficient of f which is positive, and so since the unique root of g is $\{x_0\}$ which is positive, $g(x)$ must be negative for $x < \{x_0\}$, in particular at $x = 0$. Hence

$$\lim_{x\to\infty} f_1(x) = \lim_{x\to\infty} -x^n g(1/x) \;=\; \lim_{x\to\infty} -x^n g(0) \;=\; +\infty,$$

so the leading order coefficient of f_1 is positive.