Math 104B, Number Theory, Winter 2003.

Lecture 20. Quadratic Irrationals.

A **quadratic irrational** x is an irrational number solving a quadratic equation with integer coefficients. It can be written in the form $x = a + b\sqrt{n}$ where n is an integer which is not a perfect square, a and b are rational, and $b \neq 0$. Conversely every number which can be written in this form is a quadratic irrational. The representation $a+b\sqrt{n}$ is not unique, for example $1 + 2\sqrt{3} = 1 + \sqrt{12}$. However, if $x = a + b\sqrt{n}$ is a quadratic irrational then a and $b\sqrt{n}$ are uniquely determined by x and hence the **conjugate** $a - b\sqrt{n}$ is well defined.
(To see this, suppose $a + b\sqrt{n} = c + d\sqrt{m}$ where m is not a perfect square. Then $(b\sqrt{n})^2 = ((c-a) + d\sqrt{m})^2$ is rational. But then $(c-a)d\sqrt{m}$ is rational, so $a = c$.)

By a suitable choice of n, we can write any quadratic irrational in the form

$$(*) \qquad \frac{A+\sqrt{n}}{B}, \qquad \text{where} \qquad A, B \in \mathbb{Z}, \quad B | A^2 - n.$$

For example

$$\frac{a}{b} + \frac{c}{d}\sqrt{m} = \frac{abd^2 + \sqrt{(b^2cd)^2m}}{(bd)^2}.$$

Theorem 11.4.1. If x has the form in (*) then complete quotients x_k of the continued fraction expansion of x have the form

$$\frac{A_k+\sqrt{n}}{B_k}, \qquad \text{where} \qquad A_k, B_k \in \mathbb{Z}, \quad B_k | A_k^2 - n.$$

We follow the proof in the book.

Theorem (Lagrange). The continued fraction expansion of a quadratic irrational is eventually periodic.

Before proving this, we remark that the converse of this is easy to prove. We already showed that a purely periodic continued fraction $[\overline{a_0, \ldots, a_n}]$ is a quadratic irrational. But if $x = a+b\sqrt{n}$ then $1/x = \bar{x}/(a^2-b^2n)$ is also a quadratic irrational, so a continued fraction of the form $[b_0, \ldots, b_m, \overline{a_0, \ldots, a_n}]$ is a quadratic irrational.

Proof of the Theorem. Let x be a continued fraction which we write as $(A+\sqrt{n})/B$ where A and B are integers with $B|A^2 - n$. Then write $x_k = (A_k + \sqrt{n})/B_k$ for the kth complete quotient of x. We aim to show that for k sufficiently large, the integers A_k and B_k satisfy

$$(1) \qquad 0 < A_k < \sqrt{n}, \qquad 0 < B_k < 2\sqrt{n}.$$

Since this gives only a finite number of possibilities for the pair (A_k, B_k), we must eventually have $(A_k, B_k) = (A_j, B_j)$ for some $k > j$, hence $x_k = x_j$ and the continued fraction repeats. The bounds on A_k and B_k can be obtained by showing that for k sufficiently large,

$$1 < x_k, \qquad -1 < \bar{x_k} < 0. \tag{2}$$

The first inequality is always true. We obtain the second one later, but first let's see that these imply (1). We get

$$1 < \frac{A_k + \sqrt{n}}{B_k}, \qquad -1 < \frac{A_k - \sqrt{n}}{B_k} < 0. \tag{3}$$

Subtracting the second inequality from the first gives

$$1 < \frac{2\sqrt{n}}{B_k},$$

so which implies

$$0 < B_k < 2\sqrt{n}.$$

Adding the inequalities in (3) gives

$$0 < \frac{2A_k}{B_k},$$

so $A_k > 0$ and then the second inequality in (3) gives

$$A_k - \sqrt{n} < 0,$$

so $A_k < \sqrt{n}$.

It remains to show (2). We follow the proof of 11.4.5.