Math 104B, Number Theory, Winter 2003.

Lecture 21. Quadratic Irrationals Continued.

We completed the proof of Lemma 11.4.5 given in the book, using the following modification of Lemma 11.4.4:
Lemma If x is a quadratic irrational with $x > 1$ an $\bar{x} < 0$, and we set $y = 1/\{x\}$, then $y > 1$ and $-1 < \bar{y} < 0$.
For the proof, clearly $y > 0$, but

$$y \; = \; \frac{1}{x - \lfloor x \rfloor}, \qquad \text{so} \qquad x \; = \; \frac{1}{y} + \lfloor x \rfloor\,,$$

and so

$$\frac{1}{\bar{y}} \; + \; \lfloor x \rfloor \; = \; \bar{x} \; < \; 0.$$

Then

$$\frac{1}{\bar{y}} \; < \; -\lfloor x \rfloor \; < \; -1,$$

and so $-1 < \bar{y} < 0$.

Theorem. A quadratic irrational x has purely periodic continued fraction expansion if and only if $x > 1$ and $-1 < \bar{x} < 0$.

Proof. Certainly if $x = [\overline{a_0, \ldots, a_n}]$, then the sequence of complete quotients x_k is periodic, that is $x_{k+n+1} = x_k$ for every $k \geq 0$. But eventually $x_k > 1$ and $-1 < \bar{x_k} < 0$, and so this must hold for all values of k, in particular for $x = x_0$.

Now suppose that $x > 0$ and $-1 < \bar{x} < 0$. By the lemma we proved earlier, for all complete quotients x_k we have $x_k > 0$ and $-1 < \bar{x_k} < 0$. Furthermore, since the continued fraction of x is eventually periodic, there exists i and j with $i < j$ and $x_i = x_j$. We aim to show that this implies $x_{i-1} = x_{j-1}$. If we can show this then we can iterate to get $x_{i-\ell} = x_{j-\ell}$ for every $\ell \leq i$, and so the continued fraction expansion of x is periodic.
Problem: You know x_i, how do you get x_{i-1}? Answer:

$$x_i \; = \; \frac{1}{x_{i-1} - \lfloor x_{i-1} \rfloor}, \qquad \text{so} \qquad x_{i-1} \; = \; \frac{1}{x_i} + \lfloor x_{i-1} \rfloor\,.$$

The problem is to determine $\lfloor x_{i-1} \rfloor$.

Lemma 11.5.1. If $x > 1$ is a quadratic irrational with $-1 < \bar{x} < 1$ and if $y = 1/\{x\}$, then

$$\lfloor x \rfloor \; = \; \left\lfloor \frac{-1}{\bar{y}} \right\rfloor.$$

Proof:

$$\frac{-1}{y} \;=\; \lfloor x \rfloor - \bar{x},$$

but $0 < -\bar{x} < 1$, so

$$\left\lfloor \frac{-1}{y} \right\rfloor \;=\; \lfloor x \rfloor .$$

This completes the proof of the Theorem.

Corollary. The continued fraction expansion of $\sqrt{n}$ has the form $\sqrt{n} = [a_0, \overline{a_1, \ldots, a_k}]$.

Proof. Since $\overline{\sqrt{n}} = -\sqrt{n} < 0$, by the first Lemma proved today, $-1 < x_1 < 0$, so x_1 is purely periodic.

Proposition. If $x = [\overline{a_0, \ldots, a_k}]$, then $-1/\bar{x} = [\overline{a_k, \ldots, a_0}]$.