Math 104B, Number Theory, Winter 2003.

Lecture 22. Brahmagupta-Bhaskara equation.

Proposition. If $x = [\overline{a_0, \ldots, a_k}]$, then $-1/\bar{x} = [\overline{a_k, \ldots, a_0}]$.

Proof. Consider the map T which sends a real number x to the next complete quotient in the continued fraction expansion, that is

$$T : x \to \frac{1}{\{x\}}.$$

We showed that if $x > 1$ and $-1 < \bar{x} < 0$, and if $T : x \to y$, then x can be recovered from y. Indeed,

$$\lfloor x \rfloor = \left\lfloor \frac{-1}{\bar{y}} \right\rfloor.$$

This follows since

$$\frac{-1}{\bar{y}} = \lfloor x \rfloor - \bar{x}.$$

Now we show in addition that

$$T : \frac{-1}{\bar{y}} \to \frac{-1}{\bar{x}}.$$

Indeed,

$$\frac{1}{\frac{-1}{\bar{y}} - \left\lfloor \frac{-1}{\bar{y}} \right\rfloor} = \frac{1}{\lfloor x \rfloor - \bar{x} - \lfloor x \rfloor} = \frac{-1}{\bar{x}}.$$

Writing x_k for the kth complete quotient of $x = [\overline{a_0, \ldots, a_k}]$, we have that under T,

$$x_0 \to x_1 \to x_2 \to \cdots \to x_{k-1} \to x_k \to x_0 \to x_1 \to \ldots.$$

Then

$$\frac{-1}{\bar{x}_0} \to \frac{-1}{\bar{x}_k} \to \frac{-1}{\bar{x}_{k-1}} \to \cdots \to \frac{-1}{\bar{x}_2} \to \frac{-1}{\bar{x}_1} \to \frac{-1}{\bar{x}_0} \to \ldots$$

and

$$\left\lfloor \frac{-1}{\bar{x}_0} \right\rfloor = a_k, \qquad \left\lfloor \frac{-1}{\bar{x}_k} \right\rfloor = a_{k-1}, \qquad \ldots$$

so

$$\frac{-1}{\bar{x}} = [\overline{a_k, \ldots, a_0}].$$

Now consider $\sqrt{n} = [a_0, a_1, \ldots]$. Then $\sqrt{n} + a_0 = [2a_0, a_1, a_2, \ldots]$. But $\sqrt{n} + a_0 > 1$, and $\overline{\sqrt{n} + a_0} = -\sqrt{n} + a_0$ which is between -1 and 0, so $\sqrt{n} + a_0 = [\overline{2a_0, a_1, \ldots, a_k}]$. But then

$$\frac{-1}{\overline{\sqrt{n} + a_0}} = [\overline{a_k, \ldots, a_1, 2a_0}],$$

and

$$2a_0 + \frac{1}{\frac{-1}{\overline{\sqrt{n}+a_0}}} \;=\; [\overline{2a_0, a_k, \ldots, a_1}].$$

But

$$2a_0 + \frac{1}{\frac{-1}{\overline{\sqrt{n}+a_0}}} \;=\; 2a_0 + \sqrt{n} - a_0 = \sqrt{n} + a_0 \;=\; [\overline{2a_0, a_1, \ldots, a_k}].$$

Hence

$$[\overline{2a_0, a_k, \ldots, a_1}] \;=\; [\overline{2a_0, a_1, \ldots, a_k}] \;=\; [\overline{2a_0, a_1, \ldots, a_j, \ldots, a_1}],$$

and

$$\sqrt{n} \;=\; [a_0, \overline{a_1, \ldots, a_j, \ldots, a_1, 2a_0}].$$

Diophantine Approximation. x is a real number. The rational p/q is a
(a). *good approximation* to x if

$$\left| x - \frac{p}{q} \right| \;<\; \left| x - \frac{p'}{q'} \right|$$

for every rational p'/q' with $1 \leq q' < q$.
(b). *best approximation* to x if

$$|xq - p| \;<\; |xq' - p'|$$

for every rational p'/q' with $1 \leq q' < q$.

Note that a best approximation is a good approximation for if $1 \leq q' < q$ then

$$|xq - p| \;<\; |xq' - p'| \qquad \Rightarrow \qquad \left| x - \frac{p}{q} \right| \;<\; \frac{q'}{q} \left| x - \frac{p'}{q'} \right| \;<\; \left| x - \frac{p'}{q'} \right|$$

Example. Consider $2/9$. We have

$$0 < \frac{1}{9} < \frac{1}{8} < \frac{1}{7} < \frac{1}{6} < \frac{1}{5} < \frac{2}{9} < \frac{1}{4} < \frac{2}{7} < \frac{1}{3} < \frac{2}{5} < \frac{1}{2}.$$

The good approximations to $2/9$ are

$$0,\ \frac{1}{3},\ \frac{1}{4},\ \frac{1}{5},\ \frac{2}{9}.$$

Now

$$\left| \frac{2}{9} - 0 \right| = \frac{2}{9}, \qquad \left| 3 \cdot \frac{2}{9} - 1 \right| = \frac{3}{9}, \qquad \left| 4 \cdot \frac{2}{9} - 1 \right| = \frac{1}{9}, \qquad \left| 5 \cdot \frac{2}{9} - 1 \right| = \frac{1}{9}, \qquad \left| 9 \cdot \frac{2}{9} - 2 \right| = 0.$$

The best approximations to 2/9 are

$$0, \ \frac{1}{4}, \ \frac{2}{9}.$$

Note that

$$2 = 0 \cdot 9 + 2, \qquad 9 = 4 \cdot 2 + 1 + 0, \qquad 2 = 2 \cdot 1 + 0,$$

so

$$\frac{2}{9} = [0, 4, 2]$$

and the convergents are $0, \frac{1}{4}, \frac{2}{9}$.

Theorem. The best approximations of x are precisely the convergents of the continued fraction expansion of x.

Corollary. If

$$\left| x - \frac{p}{q} \right| < \frac{1}{2q^2},$$

then p/q is a best approximation to x and so p/q is a convergent of the continued fraction expansion of x.

Proof. Suppose that $1 \le q' < q$. Then

$$\left| \frac{p'}{q'} - x \right| \ \geq \ \left| \frac{p'}{q'} - \frac{p}{q} \right| \ - \ \left| \frac{p}{q} - x \right| \ > \ \frac{|p'q - pq'|}{q'q} \ - \ \frac{1}{2q^2} \ \geq \ \frac{1}{q'q} - \frac{1}{2q'q} \ \geq \ \frac{1}{2q'q}.$$

Hence

$$|xq' - p'| \ \geq \ \frac{1}{2q} \ > \ |xq - p|.$$

Brahmagupta-Bhaskara equation.
Given an integer d which is not a perfect square, we look for solutions to $x^2 - dy^2 = 1$. We see that if (x, y) is a solution of this equation then x/y is a convergent of the partial fraction expansion of $\sqrt{d}$. Indeed, $x^2/y^2 > d$, and

$$\left| \sqrt{d} - \frac{x}{y} \right| \ = \ \frac{\left| d - \frac{x^2}{y^2} \right|}{\sqrt{d} + \frac{x}{y}} \ = \ \frac{|dy^2 - x^2|}{y^2 \left(\sqrt{d} + \frac{x}{y} \right)} \ < \ \frac{1}{2y^2\sqrt{d}} \ < \ \frac{1}{2y^2}.$$

We can apply the corollary.
Example. Find the solutions to $x^2 - 2y^2 = 1$.

Solution. $\sqrt{2} = [1, 2, 2, 2, 2, 2, \ldots]$ and the c convergents are $1, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \ldots$ and

$$3^2 - 2 \cdot 2^2 = 1, \qquad 7^2 - 2 \cdot 5^2 = -1, \qquad 17^2 - 2 \cdot 12^2 = 1, \ldots$$

We find that every other convergent gives a solution.