Math 130A. Ordinary Differential Equations

Solution of a practice final problem.
Problem. Consider the system

$$x' = y - f(x), \qquad y' = -x.$$

Find the equilibrium points and determine when the linearlized matrix is hyperbolic.

Solution. The equilibrium point is $(0, f(0))$. We have

$$D\begin{pmatrix} y - f(x) \\ -x \end{pmatrix} = \begin{pmatrix} -f'(x) & 1 \\ -1 & 0 \end{pmatrix}.$$

Evaluating at the equilibrium point we get the linearized equation

$$Y' = \begin{pmatrix} -f'(0) & 1 \\ -1 & 0 \end{pmatrix} Y.$$

The characteristic equation is

$$\left(\lambda^2 + \frac{f'(0)}{2}\right)^2 + 1 - \frac{(f'(0))^2}{4}.$$

so the roots are

$$\lambda = -\frac{f'(0)}{2} \pm \frac{\sqrt{(f'(0))^2 - 4}}{2}$$

This is hyperbolic provided $f'(0) \neq 0$. If the roots are real, then they both have the same sign as $f'(0)$. The original equation has a source if $f'(0) < 0$ and a sink if $f'(0) > 0$. We remark that we do not distinguish for example between a *spiral sink* and a *sink* because we only know the behavior of the non-linear system up to conjugacy, and the sink and spiral sink are conjugate to each other.