Homework 6 Problems.

H1. (a) Write down a list of all the possible real canonical forms for 3×3 matrices.

Set

$$A = \begin{pmatrix} 1 & -4 & 2 \\ -2 & 5 & 2 \\ -12 & -12 & 15 \end{pmatrix}.$$

The characteristic polynomial is $p(t) = (t-3)(t-9)^2$.

(b) Calculate an eigenvector V_1 for A with eigenvalue $\lambda_1 = 9$, and describe the eigenspace for $\lambda_1 = 9$.

(c) Write down the canonical form for A and explain your answer.

(d) Calculate an eigenvector V_3 of A with eigenvalue $\lambda_3 = 3$.

(e) Write down the general solution to the equation $(A - 9I)W = V_1$. Choose a particular solution $W = V_2$ satisfying this equation.

(f) Using the equations $(A - 9I)V_1 = 0$, $(A - 9I)V_2 = V_1$ and $(A - 3I)V_3 = 0$, express AV_1, AV_2 and AV_3 as linear combinations of V_1, V_2, V_3, and hence write down the matrix B of A relative to the basis V_1, V_2, V_3.

(g) Calculate a matrix T so that the following diagram commutes. ($A = TBT^{-1}$).

$$\begin{tikzcd} X \arrow[r, "A"] & AX \\ Y \arrow[u, "T"] \arrow[r, "B"] & BY \arrow[u, "T"'] \end{tikzcd}$$

(h) Calculate T^{-1}.

(i) Calculate explicitly $T^{-1}AT$.

(j) Write $B = \hat{S} + \hat{N}$ where $\hat{S}$ is diagonal and $\hat{N}$ is nilpotent. (Recall that N is nilpotent if $N^k = 0$ for some k. Any matrix with zeros on and below the diagonal is nilpotent.)

(k) Compute $\hat{N}^2$, $\hat{N}^3$, $\hat{N}^4$, and hence calculate

$$e^{t\hat{N}} = I + t\hat{N} + \frac{(t\hat{N})^2}{2} + \frac{(t\hat{N})^3}{3!} + \dots.$$

(l) Check that $(d/dt)e^{t\hat{N}} = \hat{N}e^{t\hat{N}}$ by computing both sides.

(m) Write down $e^{t\hat{S}}$.

(n) Compute $e^{tB} = e^{t\hat{S}}e^{t\hat{N}}$.

(o) Write $A = S + N$ where S is semisimple (the (complex) canonical form for S is diagonal) and N is nilpotent. (Hint: $A = T\hat{S}T^{-1} + T\hat{N}T^{-1}$.)

(p) Evaluate $e^{tA} = Te^{tB}T^{-1}$.

(q) Write the solution of $X' = AX$ with $X(0) = \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}$.

H2. Let

$$A = \begin{pmatrix} 1 & 1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(a) Calculate the (complex) eigenvalues and eigenvectors for A.

(b) Solve $X' = AX$ by using the fact that if λ is an eigenvalue of A with eigenvector V, then $e^{\lambda t}V$ is a solution to $X' = AX$ and so are its real and imaginary parts. Using this, write down the general solution.

(c) Calculate a complex matrix T and a complex diagonal matrix D such that

$$A = TDT^{-1}.$$

(e) Write down the matrix e^{tD} and separate it into its real and imaginary parts by using Euler's formula

$$e^{a+ib} = e^a(\cos b + i \sin b).$$

(f) Calculate

$$e^{tA} = Te^{tD}T^{-1}.$$

Use your answer to (f) to solve $X' = AX$, and compare with (b).

H3. Recall that if V is an n-dimensional vector space and W is a vector space and if $T : V \to W$ is a linear map, and if $R(T)$ denotes the image of T and $N(T)$ denotes the null space of T, then

$$\dim(V) = \dim(N(T)) + \dim(R(T)).$$

Suppose that A is a real 3×3 matrix with characteristic polynomial

$$p(\lambda) = \lambda^2(\beta - \lambda), \qquad \alpha \neq \beta.$$

Suppose in addition that the eigenspace for $\lambda = 0$ is spanned by the vector X and the eigenspace for $\lambda = \beta$ is spanned by the vector Y.

(a) Explain why the range $W = R(A)$ is two dimensional.

(b) Explain why AW is a subspace of W.

(c) Explain why Y is in W.

(d) Consider the operator $T : W \to W$ given by $TZ = AZ$. Notice that Y is an eigenvector of T with eigenvalue β. Suppose that the characteristic polynomial of T is $q(\lambda) = (\beta - \lambda)(\gamma - \lambda)$. What is the value of γ? Explain.

(e) Explain why $X \in W$.

(f) Take a vector Z in $\mathbb{R}^3$ with $AZ = X$. What is the basis of A with respect to the basis X, Z, Y?