Math 130A. Ordinary Differential Equations

Lecture 1: First Order Equations.

We study equations of the form:

(*) $$\frac{dx}{dt} = f(x,t), \qquad t \geq 0.$$

A solution is a function $u(t) = x$ so that

$$u'(t) = f(u(t), t)$$

at every point t.
An **initial value problem** is where we study the equation (*) with an initial condition

$$x(t_0) = x_0.$$

Example 1.

(1) $$\frac{dx}{dt} = \cos t.$$

Solution by integration.

$$x(t) = \sin t + c,$$

where c is a constant.
Clearly any choice of c gives a solution.
Claim: Every solution has this form. To see this, we assume we know the theorem that **a differentiable function whose derivative vanishes everywhere must be constant**. (140A).
Now suppose $u(t)$ satisfies (1). Then

$$\frac{d(u(t) - \sin t)}{dt} = \frac{du}{dt} - \cos t = \cos t - \cos t = 0.$$

Hence

$$u(t) - \sin t = c, \qquad u(t) = \sin t + c.$$

Remark. The value of c is fixed by the value of x at any given point t_0. Indeed, if we specify

$$x(t_0) = x_0,$$

then

$$x_0 = x(t_0) = \sin t_0 + c,$$

so

$$c = x_0 - \sin t_0.$$

Theorem. *If f is a continuous function on $[t_0, \infty)$ and x_0 is a real number, then the unique solution to*

$$\frac{dx}{dt} = f(t), \qquad x(t_0) = x_0$$

is

$$x(t) = x_0 + \int_{t_0}^{t} f(\tau)\, d\tau.$$

An **autonomous** first order differential equation is a differential equation of the form

$$(**) \qquad \frac{dx}{dt} = f(x)$$

For the rest of the lecture we will study such equations. We remark that they can be integrated by writing them as

$$\frac{dt}{dx} = \frac{1}{f(x)},$$

so

$$t(x) = t_0 + \int_{x(0)}^{x} \frac{1}{f(\xi)}\, d\xi.$$

Although this is useful to solve some homework problems, we will not want to think of the autonomous equations in these terms right now.

Translation invariance. Suppose $u(t)$ is a solution to (**). Then so is $v(t) = u(t - t_0)$. Indeed,

$$\frac{dv}{dt}(t) = \frac{du}{dt}(t - t_0) = f(u(t - t_0)) = f(v(t)).$$

Hence if we wish to solve (**) with $x(t_0) = u_0$, we might as well assume $t_0 = 0$.

Example 2.

$$(2) \qquad \frac{dx}{dt} = ax.$$

Here a is a constant. This gives a simple model of population growth. The rate at which the population grows is proportional to the current size of the population. We can check that a solution is given by

$$(3) \qquad x(t) = ke^{at}$$

where k is a constant. To derive this, we have

$$\frac{dt}{dx} = \frac{1}{ax} \Rightarrow t = \frac{1}{a}(\log|x| + c) \Rightarrow |x| = e^{-c}e^{at} \Rightarrow x = (\pm e^{-c})e^{at}.$$

In fact we can show directly that any solution must have the form (3). Indeed, suppose that $u(t)$ is a solution to (2). Then

$$\frac{d(e^{-at}u)}{dt} = -ae^{-at}u + e^{-at}\frac{du}{dt} = -ae^{-at}u(t) + ae^{at}u(t) = 0.$$

Hence

$$e^{-at}u(t) = k,$$

and

$$u(t) = ke^{at}.$$

Notice that $k = u(0)$, so

$$u(t) = u(0)e^{at}.$$

Conclusion: The unique solution to $x' = ax$ with $x(0) = x_0$ is $x = x_0e^{at}$.

Principle. The **phase line** is the x-axis divided into those intervals where f is positive and negative. Each interval where f is positive is labeled with an upward arrow. Each interval where f is negative is labeled with a downward arrow. Assume that $f(x)$ is a differentiable function on $\mathbb{R}$, and x_0 is a point on the phase line. We seek a solution of the equation

$$x' = f(x), \qquad x(0) = x_0.$$

If $f(x_0) = 0$, the unique solution is the constant $u(t) \equiv x_0$.
If $f(x_0) \neq 0$, there is a solution $u(t)$ on $[0, T)$ such that as t increases from zero, the value of $u(t)$ starts at x_0 and travels in the direction of the arrow on the phase line, tending towards the endpoint of the interval as $t \to T$. If the end of the interval is finite, then $T = \infty$.