

Lecture 10: Examples.

Example 1. For the matrix

$$A = \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix},$$

- (a) Find the eigenvalues and eigenvectors.
- (b) Find the matrix T which puts A in canonical form.
- (c) Find the general solution of $X' = AX$ and $Y' = (T^{-1}AT)Y$.
- (d) Sketch the phase portraits of both systems.
- (e) Name the type of equilibrium which occurs.

Solution.

(a). After a standard computation, the eigenvalues are $\frac{1}{2} \pm \frac{\sqrt{3}}{2}i$. The eigenvector for $\frac{1}{2} + \frac{\sqrt{3}}{2}i$ is $V_1 + iV_2$,

$$V_1 + iV_2 = \begin{pmatrix} -1 \\ \frac{1}{2} - \frac{\sqrt{3}}{2}i \end{pmatrix}.$$

The eigenvector for the conjugate eigenvalue $\frac{1}{2} - \frac{\sqrt{3}}{2}i$ is the conjugate eigenvector

$$V_1 - iV_2 = \begin{pmatrix} -1 \\ \frac{1}{2} + \frac{\sqrt{3}}{2}i \end{pmatrix}.$$

(b) **Fact to remember:** The matrix T which puts A in canonical form is the change of basis matrix $[V_1, V_2]$ from the standard basis to the basis V_1, V_2 .

In our case, this is

$$T = [V_1, V_2] = \begin{pmatrix} -1 & 0 \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix}.$$

Then the canonical form for A is $T^{-1}AT$. What is this canonical form? It is obtained by computing

$$(*) \quad A(V_1 + iV_2) = (\alpha + i\beta)(V_1 + iV_2) = \alpha V_1 - \beta V_2 + i(\beta V_1 + \alpha V_2).$$

The canonical form for A is the matrix of the linear map “multiplication by A ”, relative to the new basis V_1, V_2 . We read off from (*) that it is $\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$, which in our case now is

$$T^{-1}AT = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}.$$

We checked in lecture that this is correct by computing the product

$$T^{-1}AT = \frac{2}{\sqrt{3}} \begin{pmatrix} -\frac{\sqrt{3}}{2} & 0 \\ -\frac{1}{2} & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ \frac{1}{2} & -\frac{\sqrt{3}}{2} \end{pmatrix} = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & \frac{1}{2} \end{pmatrix}$$

Remark. We note that in the general case, if we work in coordinates relative to the basis V_1, V_2 , then A takes the canonical form $\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$. In the new coordinates relative to V_1, V_2 , the vector V_1 corresponds to $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and the vector V_2 corresponds to $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Consequently the eigenvector $V_1 + iV_2$ corresponds to $\begin{pmatrix} 1 \\ i \end{pmatrix}$. You can check that this is indeed an eigenvector for the canonical matrix

(c) For the equation

$$Y' = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} Y,$$

we have the eigenvector $\begin{pmatrix} 1 \\ i \end{pmatrix}$ with eigenvalue $\alpha + i\beta$ and so we can write down the solution

$$Y(t) = e^{(\alpha+i\beta)t} \begin{pmatrix} 1 \\ i \end{pmatrix}.$$

Taking real and imaginary parts by using Euler's formula

$$e^{i\theta} = \cos \theta + i \sin \theta,$$

we get two linearly independent solutions

$$Y_1(t) = e^{\alpha t} \begin{pmatrix} \cos \beta t \\ -\sin \beta t \end{pmatrix}, \quad Y_2(t) = e^{\alpha t} \begin{pmatrix} \sin \beta t \\ \cos \beta t \end{pmatrix}.$$

The general solution is a linear combination of these:

$$Y(t) = c_1 Y_1(t) + c_2 Y_2(t).$$

In our case, this becomes

$$Y(t) = e^{t/2} \begin{pmatrix} c_1 \cos \sqrt{3}t/2 + c_2 \sin \sqrt{3}t/2 \\ -c_1 \sin \sqrt{3}t/2 + c_2 \cos \sqrt{3}t/2 \end{pmatrix}.$$

For the original equation $X' = AX$, if we have eigenvector $V_1 + iV_2$ with eigenvalue $\alpha + i\beta$ then a complex solution is given by

$$e^{(\alpha+i\beta)t}(V_1 + iV_2) = e^{\alpha t} (\cos \beta t V_1 - \sin \beta t V_2 + i(\sin \beta t V_1 + \cos \beta t V_2)).$$

The general solution obtained from taking real and imaginary parts and forming a linear combination is then

$$\begin{aligned} X(t) &= e^{\alpha t} (c_1 (\cos \beta t V_1 - \sin \beta t V_2) + c_2 (\sin \beta t V_1 + \cos \beta t V_2)) \\ &= e^{\alpha t} (c_1 \cos \beta t + c_2 \sin \beta t) V_1 + (-c_1 \sin \beta t + c_2 \cos \beta t) V_2 \end{aligned}$$

In our case, this becomes

$$X(t) = e^{t/2} \left(c_1 \cos \sqrt{3}t/2 + c_2 \sin \sqrt{3}t/2 \right) \begin{pmatrix} -1 \\ 1/2 \end{pmatrix} + e^{t/2} \left(-c_1 \sin \sqrt{3}t/2 + c_2 \cos \sqrt{3}t/2 \right) \begin{pmatrix} 0 \\ -\sqrt{3}/2 \end{pmatrix}.$$

(d) We omit the pictures in these notes. The solution curves are spiraling outwards as t increases. The solution $Y(t)$ spirals out clockwise. Indeed, in the general case, the vectors

$$\begin{pmatrix} \cos \beta t \\ -\sin \beta t \end{pmatrix}, \quad \begin{pmatrix} \sin \beta t \\ \cos \beta t \end{pmatrix}$$

circle around clockwise when $\beta > 0$. Our solution is a fixed linear combination of these scaled up by $e^{\alpha t}$.

In the case of $X(t)$, we can find the direction in which the general solution spirals by considering just the special case where $c_2 = 0$ or $c_1 = 0$. Indeed, consider the case $c_1 = 0$. The solution in this case (for arbitrary β) is $e^{\alpha t}(\sin \beta t V_1 + \cos \beta t V_2)$. Let's ignore the scale factor $e^{\alpha t}$. We see that the vector $(\sin \beta t V_1 + \cos \beta t V_2)$ starts at V_1 at time $t = 0$ and has velocity βV_2 . In our case, $\beta > 0$.

Question: In which direction are you spiraling around the origin if you start at $V_1 = \begin{pmatrix} -1 \\ 1/2 \end{pmatrix}$ and move with initial velocity which is a positive multiple of $V_2 = \begin{pmatrix} 0 \\ -\sqrt{3}/2 \end{pmatrix}$? A simple sketch of V_1 and V_2 shows that you are traveling counterclockwise.

(e). Spiral Source.

Example 2. Find the general solution of $x'' - x' - x = 0$, and sketch the phase portrait in the (x, x') plane. Describe how $x(t)$ behaves as $t \rightarrow \infty$.

Solution. We set $y = x'$ to put the equation in the form

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}.$$

The eigenvalues and eigenvectors are

$$\lambda_1 = \frac{1 + \sqrt{3}}{2}, \quad V_1 = \begin{pmatrix} 1 \\ (1 + \sqrt{3})/2 \end{pmatrix}, \quad \lambda_2 = \frac{1 - \sqrt{3}}{2}, \quad V_2 = \begin{pmatrix} 1 \\ (1 - \sqrt{3})/2 \end{pmatrix}.$$

In general, two linearly independent solutions are

$$X_1(t) = e^{\lambda_1 t} V_1, \quad X_2(t) = e^{\lambda_2 t} V_2.$$

The general solution is a linear combination of these,

$$X(t) = c_1 X_1(t) + c_2 X_2(t).$$

In our case, this is

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = c_1 e^{(1+\sqrt{3})t/2} \begin{pmatrix} 1 \\ (1 + \sqrt{3})/2 \end{pmatrix} + c_2 e^{(1-\sqrt{3})t/2} \begin{pmatrix} 1 \\ (1 - \sqrt{3})/2 \end{pmatrix}.$$

Since we have $\lambda_2 = (1 - \sqrt{3})/2 < 0$, we see that $e^{\lambda_2 t} \rightarrow 0$ as $t \rightarrow \infty$, and

$$x(t) - c_1 e^{(1+\sqrt{3})t/2} \rightarrow 0 \quad \text{as } t \rightarrow \infty.$$

Hence the solution $x(t)$ approaches the solution $z(t) := c_1 e^{(1+\sqrt{3})t/2}$ as $t \rightarrow \infty$.

Remark. Since the error $x(t) - z(t)$ is exponentially decreasing as $t \rightarrow \infty$, in fact $x(t)$ approaches $z(t)$ very fast as $t \rightarrow \infty$, and the derivatives of $x(t)$ approach those of $z(t)$ as well.