MATH 130A. ORDINARY DIFFERENTIAL EQUATIONS

Lecture 13: Three dimensional linear systems.

Recall. For the 2×2 system $X' = AX$: Suppose that A has eigenvalues $\lambda_1 < 0 < \lambda_2$ and eigenvectors V_1 and V_2. The general solution is

$$X(t) \;=\; c_1 e^{\lambda_1 t} V_1 \;+\; c_2 e^{\lambda_2 t} V_2.$$

The equilibrium point $X = 0$ is a saddle. Notice that if $c_2 = 0$, then the solution $X(t)$ stays on the line spanned by V_1, and converges to zero as $t \to \infty$. This line consisting of all the points $c_1 V_1$ with c_1 a real number, is called the **stable line**, because the solution converges to the equilibrium as $t \to \infty$. On the other hand, the line $c_2 V_2$ with c_2 a real number is called the **unstable line**.

3×3 **linear systems**. Now we wish to solve the system $X' = AX$ where

$$X \;=\; \begin{pmatrix} x \\ y \\ z \end{pmatrix}, \qquad A \;=\; \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}.$$

Let us suppose that A has real eigenvalues $\lambda_1, \lambda_2, \lambda_3$ with eigenvectors V_1, V_2, V_3. Then it is easy to check that the general solution is

$$\begin{aligned} X(t) \;&=\; c_1 e^{\lambda_1 t} V_1 \;+\; c_2 e^{\lambda_2 t} V_2 \;+\; c_3 e^{\lambda_3 t} V_3 \\ &=\; [V_1, V_2, V_3] \begin{pmatrix} e^{t\lambda_1} & 0 & 0 \\ 0 & e^{t\lambda_2} & 0 \\ 0 & 0 & e^{t\lambda_3} \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix} \;=\; TD(t)Y_0, \end{aligned}$$

where c_1, c_2, c_3 are constants, and

$$Y_0 = \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}, \qquad D(t) = \begin{pmatrix} e^{t\lambda_1} & 0 & 0 \\ 0 & e^{t\lambda_2} & 0 \\ 0 & 0 & e^{t\lambda_3} \end{pmatrix},$$

and $T = [V_1, V_2, V_3]$ is the change of basis matrix to the basis of eigenvectors. We now want to give a very simple formula for the solution. Recall that the solution of the one-variable equation $x' = \lambda x$ is $x(t) = e^{\lambda t} x(0)$. Now set

$$D \;=\; \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{pmatrix}.$$

Then $D(t) = e^{tD}$. The solution to $Y' = DY$ is $Y(t) = e^{tD} Y(0)$. Now we have just seen that the solution to $X' = AX$ is $X(t) = Te^{tD} Y_0$ for some constant vector Y_0. We will see that in fact this has the form

$$(*) \qquad X(t) \;=\; e^{tA} X(0).$$

Indeed, we have $A = TDT^{-1}$, and by the power series definition of the exponential we have

$$\begin{aligned} e^{tA} &= \sum_{k=0}^{\infty} \frac{(tA)^k}{k!} = \sum_{k=0}^{\infty} \frac{t^k (TDT^{-1})^k}{k!} \\ &= \sum_{k=0}^{\infty} \frac{t^k T D^k T^{-1}}{k!} = T \sum_{k=0}^{\infty} \frac{(tD)^k}{k!} T^{-1} = T e^{tD} T^{-1}. \end{aligned}$$

Hence

$$X(t) = T e^{tD} Y_0 = e^{tA}(TY_0).$$

But setting $t = 0$ in this we get (*).

We worked through the example on page 108 and defined stable and unstable planes and stable lines.

We worked through the example on page 111 and defined a spiral center.

Normal forms for 3×3 matrices:

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \mu & 0 \\ 0 & 0 & \nu \end{pmatrix}, \qquad \lambda, \mu, \nu \textbf{ real}.$$

$$\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \mu \end{pmatrix}, \qquad \lambda, \mu \textbf{ real}.$$

$$\begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}, \qquad \lambda \textbf{ real}.$$

$$\begin{pmatrix} \alpha & \beta & 0 \\ -\beta & \alpha & 0 \\ 0 & 0 & \lambda \end{pmatrix}, \qquad \alpha, \beta, \lambda \textbf{ real}.$$