MATH 130A. ORDINARY DIFFERENTIAL EQUATIONS

Lectures 18 and 19: 3×3 **Linear Algebra**.

There are four canonical forms for a real 3×3 matrix. If $\lambda_1, \lambda_2, \lambda_3, \alpha, \beta$ are arbitrary real numbers, then these are

$$\text{(a)} \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{pmatrix}, \qquad \text{(b)} \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & 1 \\ & & \lambda_2 \end{pmatrix}, \qquad \text{(c)} \begin{pmatrix} \lambda_1 & 1 & \\ & \lambda_1 & 1 \\ & & \lambda_1 \end{pmatrix},$$

$$\text{(d)} \begin{pmatrix} \lambda_1 & & \\ & \alpha & \beta \\ & -\beta & \alpha \end{pmatrix}.$$

If A is a real 3×3 matrix, there are 4 possible types of characteristic polynomial $p(\lambda) = \det(A - \lambda I)$:

Case 1: $p(\lambda) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)(\lambda_3 - \lambda)$ where $\lambda_1, \lambda_2, \lambda_3$ are distinct and real.

Case 2: $p(\lambda) = (\lambda_1 - \lambda)(\lambda_2 - \lambda)^2$, where $\lambda_1 \neq \lambda_2$.

Case 3: $p(\lambda) = (\lambda_1 - \lambda)^3$.
Case 4: $p(\lambda) = (\alpha + i\beta - \lambda)(\alpha - i\beta - \lambda)(\lambda_1 - \lambda)$, where $\alpha,\ \beta,\ \lambda_1$ are real.

Notice that if B is the canonical form for A, then the characteristic polynomial of A is the same as that of B. Indeed, if $A = TBT^{-1}$, then

$$\begin{aligned} p(\lambda) &= \det(A - \lambda I) = \det(TBT^{-1} - \lambda I) = \det(TBT^{-1} - \lambda TT^{-1}) \\ &= \det\big(T(B - \lambda I)T^{-1}\big) = \det T \det(B - \lambda I) \det T^{-1} = \det(B - \lambda I). \end{aligned}$$

Definition. We say that U, V, W is a **canonical basis** for A, if the matrix of the linear map A with respect to this basis is canonical (i.e. of type (a),(b),(c) or (d)).

Case 1. Must be type (a). Each eigenvalue λ_j has an eigenvector V_j. A canonical basis is V_1, V_2, V_3.

Case 4. Must be type (d). The eigenvalue λ_1 has eigenvector V_1. The complex eigenvalue $\alpha + i\beta$ has complex eigenvector $V_2 + iV_3$. Then V_1, V_2, V_3 is a canonical basis.

Comment. "Most" matrices have distinct eigenvalues (which may be real or complex). Repeated eigenvalues are unusual!!!

Case 2. This can be either type (a) or (b). The canonical forms are

$$\text{(a) } B = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_2 \end{pmatrix}, \qquad \text{(b) } B = \begin{pmatrix} \lambda_1 & & \\ & \lambda_2 & 1 \\ & & \lambda_2 \end{pmatrix}.$$

For the canonical form B, consider $B - \lambda_2 I$.

$$\text{(a) } B - \lambda_2 I = \begin{pmatrix} \lambda_1 - \lambda_2 & & \\ & 0 & \\ & & 0 \end{pmatrix}, \qquad \text{(b) } B - \lambda_2 I = \begin{pmatrix} \lambda_1 - \lambda_2 & & \\ & 0 & 1 \\ & & 0 \end{pmatrix}.$$

What happens to the standard basis e_1, e_2, e_3 when multiplied by $B - \lambda_2 I$? In both cases (a) and (b), we see that $e_1 \mapsto (\lambda_1 - \lambda_2)e_1$, and

$$\text{(a)} \qquad \begin{aligned} e_2 &\mapsto 0 \\ e_3 &\to 0. \end{aligned}$$

$$\text{(b)} \qquad e_3 \mapsto e_2 \mapsto 0.$$

We can tell whether we are in case (a) or case (b) by the dimension d of the eigenspace for λ_2. Indeed, $d = 2$ in case (a) and $d = 1$ in case (b).

In case (a), we see that V_2 and V_3 can be chosen to be any base for the eigenspace of λ_2.

In case (b), we have a choice of either finding e_2 first or chosing e_3 first.
Finding e_2 first: e_2 is in the eigenspace for λ_2, and e_3 solves $(B - \lambda_2 I)e_3 = e_2$.
Finding e_3 first: e_3 is any vector in the null space of $(B - \lambda_2 I)^2$ which is not in the null space of $B - \lambda_2 I$. We set $e_2 = (B - \lambda_2 I)e_3$.

The same strategy works for finding V_1, V_2, V_3 when B is replaced by A.

Case 3. This can be either type (a) or (b) or (c). The canonical forms are

$$\text{(a) } B = \begin{pmatrix} \lambda_1 & & \\ & \lambda_1 & \\ & & \lambda_1 \end{pmatrix}, \quad \text{(b) } B = \begin{pmatrix} \lambda_1 & & \\ & \lambda_1 & 1 \\ & & \lambda_1 \end{pmatrix}, \quad \text{(c) } B = \begin{pmatrix} \lambda_1 & 1 & \\ & \lambda_1 & 1 \\ & & \lambda_1 \end{pmatrix}.$$

For the canonical form B, consider $B - \lambda_1 I$.

$$\text{(a) } B = \begin{pmatrix} 0 & & \\ & 0 & \\ & & 0 \end{pmatrix}, \quad \text{(b) } B = \begin{pmatrix} 0 & & \\ & 0 & 1 \\ & & 0 \end{pmatrix}, \quad \text{(c) } B = \begin{pmatrix} 0 & 1 & \\ & 0 & 1 \\ & & 0 \end{pmatrix}.$$

What happens to the standard basis e_1, e_2, e_3 when multiplied by $B - \lambda_1 I$? In case (a), we see that the eigenspace for λ_1 is all of $\mathbb{R}^3$, and so there exists a basis of eigenvectors.

$$\text{(b)} \qquad \begin{aligned} e_1 &\mapsto 0 \\ e_3 \mapsto e_2 &\to 0. \end{aligned}$$

$$\text{(c)} \qquad e_3 \mapsto e_2 \mapsto e_1 \mapsto 0.$$

We can see we are in case (a), (b), (c) when the eigenspace for λ_1 has dimension 3,2,1 respectively. In case (b), we can either find e_2 first or e_3 first.

Finding e_2 first: We see that e_2 is in the column space (range) of $B - \lambda_1 I$, which is one dimensional because the dimension of the null space plus the dimension of the range equals 3. Notice that e_2 is also in the null space of $B - \lambda_1 I$. We pick e_3 to solve $(B - \lambda_1 I)e_3 = e_2$, and then we pick e_1 to be any eigenvector for λ_1 which is linearly independent from e_2.

Finding e_3 first: We pick e_3 to be any vector which is not in the eigenspace for λ_1, and set $e_2 = (B - \lambda_1 I)e_3$. We then pick e_1 to be any eigenvector for λ_1 which is linearly independent from e_2.

Example. Find the general solution to $X' = AX$ where

$$A = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix}.$$

Solution. There are two methods. One is to write the vector equation

$$\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \begin{pmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

as three equations. Solving for z then y then x gives

$$\begin{aligned} x(t) &= e^{\lambda t}\left(x(0) + y(0)t + \frac{z(0)}{2}t^2\right) \\ y(t) &= e^{\lambda t}\left(x(0) + y(0)t\right) \\ z(t) &= e^{\lambda t} z(0). \end{aligned}$$

The other is to write the solution as

$$X(t) = e^{tA} X(0).$$

But writing

$$A = S + N = \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

clearly S and N commute, and

$$\begin{aligned} e^{tA} = e^{tS+tN} = e^{tS}e^{tN} &= e^{\lambda t}\left(I + tN + \frac{t^2N^2}{2!} + \frac{t^3N^3}{3!} + \dots\right) \\ &= e^{\lambda t}\begin{pmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix}. \end{aligned}$$

Hence

$$X(t) \;=\; e^{tS}\begin{pmatrix}1 & t & t^2/2\\ 0 & 1 & t\\ 0 & 0 & 1\end{pmatrix}\begin{pmatrix}x(0)\\ y(0)\\ z(0)\end{pmatrix}.$$

This is clearly the same solution.