

Lecture 2: Phase Lines.

Last time: An **autonomous** first order differential equation is a differential equation of the form

$$\frac{dx}{dt} = f(x).$$

The unique solution to $x' = ax$ with $x(0) = x_0$ is $x = x_0e^{at}$.

Behavior as $t \rightarrow \infty$: Notice that

$$\begin{aligned} a > 0 &\Rightarrow \lim_{t \rightarrow \infty} x_0e^{at} = \begin{cases} \infty & x_0 > 0 \\ 0 & x_0 = 0 \\ -\infty & x_0 < 0. \end{cases} \\ a = 0 &\Rightarrow x_0e^{at} \equiv x_0. \\ a < 0 &\Rightarrow \lim_{t \rightarrow \infty} x_0e^{at} = 0. \end{aligned}$$

Let us see how we can read this off without computing the solution.

The **phase line** is the x -axis divided into those intervals where f is positive and negative. Each interval where f is positive is labeled with an upward arrow. Each interval where f is negative is labeled with a downward arrow.

Principle. Assume that $f(x)$ is a differentiable function on \mathbb{R} , and x_0 is a point on the phase line. We seek a solution of the equation

$$x' = f(x), \quad x(0) = x_0.$$

If $f(x_0) = 0$, the unique solution is the constant $u(t) \equiv x_0$.

If $f(x_0) \neq 0$, the solution $u(t)$ exists on an interval $[0, T)$. As t increases from zero, the value of $u(t)$ starts at x_0 and travels in the direction of the arrow on the phase line, tending towards the endpoint of the interval as $t \rightarrow T$. If the end of the interval is finite, then $T = \infty$. If the endpoint of the interval is $\pm\infty$ then $T = \infty$ if and only if $\int^{\pm\infty} 1/f(x) dx$ is infinite.

Notation.

Equilibrium points. Note that if $f(x_0) = 0$, then the constant function $u(t) \equiv x_0$ is a solution. (There may also be other solutions with $x(0) = x_0$ if f is not differentiable.) A constant solution is called an **equilibrium solution** or **equilibrium point**.

Source. An equilibrium point x_0 is called a **source** if nearby solutions tend away from it. This is the case if the arrows on the phase line are pointing away from the point. If f is differentiable at x_0 , this happens if $(df/dx)(x_0) > 0$.

Sink. An equilibrium point x_0 is called a **sink** if nearby solutions tend towards it. This is the case if the arrows on the phase line are pointing towards the point. If f is differentiable at x_0 , this happens if $(df/dx)(x_0) < 0$.

Logistic Population Model (1830s).

The equation $x' = ax$ does not take into account the fact that when a population becomes large then due to various factors such as competition for food and space slow down the growth. The **logistic population growth model** which takes this into account is

$$x' = ax \left(1 - \frac{x}{N}\right).$$

Here, a and N are positive parameters. The value of a determines the growth when x is small, whereas N represents some sort of ideal size of the population above which it tends to decrease. We can normalize by setting $\tilde{x} = x/N$ so that we can assume $N = 1$, and so $\tilde{x}(t)$ now represents the ratio of the fraction of the ideal population present at time t . The equation (after renaming \tilde{x} to be x) is

$$(1) \quad x' = f_a(x) = ax(1 - x).$$

Notice that $f_a(x)$ is a differentiable function. By drawing the phase line, we make the following predictions about the solution $u(t)$:

$$\begin{aligned} a > 0 & \Rightarrow \begin{cases} \lim_{t \rightarrow \infty} u(t) = 1 & u(0) > 0 \\ u(t) \equiv 0 & u(0) = 0 \\ \lim_{t \rightarrow T} u(t) = -\infty & (T < \infty) \quad u(0) < 0 \end{cases} \\ a = 0 & \Rightarrow u(t) \equiv u(0). \\ a < 0 & \Rightarrow \begin{cases} \lim_{t \rightarrow T} u(t) = \infty & (T < \infty) \quad u(0) > 1 \\ u(t) \equiv 1 & u(0) = 1. \\ \lim_{t \rightarrow \infty} u(t) = 0 & u(0) < 1 \end{cases} \end{aligned}$$

The fact that $T < \infty$ comes from the computation

$$\begin{aligned} T &= \int_{x(0)}^{-\infty} \frac{dt}{dx} dx = \int_{x(0)}^{-\infty} \frac{1}{ax(1-x)} dx = - \int_{-x(0)}^{\infty} \frac{1}{a(-y)(1+y)} dy \\ &= \int_{-x(0)}^{\infty} \frac{1}{ay(1+y)} dy < \int_{-x(0)}^{\infty} \frac{1}{ay^2} dy < \infty. \end{aligned}$$