Math 130A. Ordinary Differential Equations

Lecture 3: Bifurcation.

Last time: **Example 2**: Logistic Population Model.

$$x' = f_a(x) = ax(1-x). \tag{1}$$

We can solve this:

$$a\,dt = \frac{dx}{x(1-x)} = \int \left(\frac{1}{x} + \frac{1}{1-x}\right) dx = \log \frac{x}{1-x} + c.$$

Hence

$$Ke^{at} = \frac{x}{1-x} = \frac{1}{1-x} - 1, \tag{2}$$

and so

$$x(t) = \frac{Ke^{at}}{1+Ke^{at}}. \tag{3}$$

Plugging $t=0$ into (2) gives

$$K = \frac{x(0)}{1-x(0)}.$$

$$x(t) = \frac{x(0)e^{at}}{1-x(0)+x(0)e^{at}}.$$

For example, if $a > 0$ and $x(0) = -1$ we get

$$\frac{-e^{at}}{2-e^{at}}.$$

This blows up in finite time as predicted. If $a < 0$ and $x(0) = 2$ we get

$$\frac{2e^{at}}{-1+2e^{at}}$$

which also blows up in finite time as predicted. The case which corresponds to the population model is $a > 0$ and $x(0) > 0$. In this case $x(t) \to 1$ as $t \to \infty$ as predicted.

The equilibrium point: When $a > 0$, we see that the equilibrium point $x = 1$ is a sink. This fact can also be determined by computing $f'(1)$.

$$f(x) = ax(1-x) \qquad \Rightarrow \qquad f'(x) = a(1-2x) \qquad \Rightarrow \qquad f'(1) = -a < 0.$$

This shows that $x = 1$ is a sink.

Example 3.

$$x' = g(x) = x - x^3 = x(1 - x^2) = x(1 + x)(1 - x).$$

Three equilibrium points $x = 0, \pm 1$.

$$g'(x) = 1 - 3x^2 = \begin{cases} 1 & x = 0 \\ -2 & x = \pm 1 \end{cases}$$

Hence 0 is a source and ± 1 are sinks.

Example 4.

$$x' = g_a(x) = x^2 - ax = x(x - a).$$

Two equilibrium points $x = 0$ and $x = a$.

$$g'(x) = 2x - a = \begin{cases} -a & x = 0 \\ a & x = a \end{cases}$$

If $a > 0$ then $x = 0$ is a sink and $x = a$ is a source. If $a < 0$ then $x = 0$ is a source and $x = a$ is a sink. If $a = 0$ then $x = 0$ is neither a source or a sink. We can put together the phase lines into a bifurcation diagram. The value $a = 0$ is a **bifurcation point** where the qualitative behavior of the solutions change. For this particular family of equations, the source and sink "pass through each other".

Bifurcation diagram. For a one-parameter family of autonomous equations

$$x' = f_a(x),$$

with parameter a, the bifurcation diagram is a 2 dimensional diagram in the (a, x) plane, where the phase line for the function f_α is placed at the line $a = \alpha$.

Example 5. We take the logistic equation but now assume that the population is harvested at a constant rate (think if fish for example). This gives the equation

$$x' = f_h(x), \qquad f_h(x) = x(1 - x) - h.$$

Now

$$x(1 - x) - h = -(x^2 - x + h) = -\left(x - \frac{1}{2} - \sqrt{\frac{1}{4} - h}\right)\left(x - \frac{1}{2} + \sqrt{\frac{1}{4} - h}\right).$$

We get two roots

$$x_\ell = \frac{1}{2} - \sqrt{\frac{1}{4} - h} \qquad x_r = \frac{1}{2} + \sqrt{\frac{1}{4} - h},$$

provided $h < 1/4$, one root if $h = 1/4$ and no roots for $h > 1/4$. We compute

$$f_h'(x) = 1 - 2x, \qquad f_h'(x_\ell) = \sqrt{1 - 2h}, \qquad f_h'(x_r) = -\sqrt{1 - 2h}.$$

We see that if $h < 1/4$ then the population becomes extinct if $x(0) < x_\ell$ and tends to an equilibrium if $x(0) > x_\ell$. However if $h \geq 1/4$ then the population becomes extinct whatever the initial population.