Math 130A. Ordinary Differential Equations

Lecture 4: Poincare Map.

Last time: Logistic Population Model with constant harvesting:

$$x' \;=\; x(1-x) - h.$$

If $h \geq 1/4$ then the population becomes extinct whatever the initial population. If $0 \leq h < 1/4$, then the population converges to a positive equilibrium provided $x(0)$ is above a certain critical value.

Example. Periodic harvesting. Consider the situation when a population, say of fish, is harvested seasonally. We introduce a periodic behavior into the harvesting rate:

$$(*) \qquad x' \;=\; f(t,x) \;=\; ax(1-x) \;-\; h(1+\sin(2\pi t)).$$

Note that the function $1+\sin(2\pi t)$ is periodic with period $t = 1$. It takes its minimum value zero at $t = 3/4+n$ where n is an integer, and its maximum value 2 at $t = 1/4+n$ where n is an integer. The rate of harvesting varies between $-2h$ and 0.

Goal: To describe the fate of the population as $t \to \infty$.

Lemma. *Suppose that $u(t)$ and $v(t)$ solve*

$$u' = f(t,u), \qquad\qquad v' = g(t,v)$$

and suppose that $g(t,x) \leq f(t,x)$ for all values of t,x. Suppose also that $u(0) = v(0)$. Then for all t,

$$v(t) \;\leq\; u(t).$$

By the Lemma, comparing the equation with periodic harvesting to the equation without harvesting, we find that if $x(t)$ is a solution to (*) then

$$(**) \qquad x(t) \;\leq\; \frac{x(0)e^{at}}{1-x(0)+x(0)e^{at}}.$$

Remark. Although in the equation the population $x(t)$ can become negative, of course this is not physically meaningful. However we claim that once $x(t)$ drops to zero or below, it cannot recover to a positive value later on. Hence if one has a solution with $x(T) > 0$, then it must be the case that $x(t)$ is positive on the interval $[0,T]$. To see the claim, just note that x' is a sum of two terms, the harvesting term $-h(1+\sin(2\pi t))$ is always negative except at isolated points where it vanishes, and the term $ax(1-x)$ is negative once x is negative. Once $x(t)$ becomes negative then it blows down to $-\infty$ in finite time by (**).

Remark. Although a solution to (*) might not be translation invariant because the equation (*) is not autonomous, the fact that $f(t,x)$ is periodic with period 1 means that we can translate solutions by integer amounts. More precisely, if $x(t)$ is a solution to (*) on $[0,n]$ and k is a positive integer, then

$$y(t) \ := \ x(t+k), \qquad\qquad 0 \le t \le n-k$$

is also a solution to (*). Indeed,

$$y'(t) \ = \ x'(t+k) \ = \ f(t+k, x(t+k)) \ = \ f(t, x(t+k)) \ = \ f(t, y(t)).$$

In particular, if $k \le n-1$, then y is a solution to (*) on $[0,1]$. We see that

$$y(0) = x(k), \qquad\qquad y(1) = x(k+1).$$

Definition. The map p from the real numbers to the real numbers which takes the initial value $x(0)$ of the solution to (*) to the value $x(1)$ at time $t=1$ is known as the **Poincare** map. We see that

$$x(1) \ = \ p(x(0)), \qquad x(2) = p(x(1)) = p(p(x(0)), \qquad x(3) = p(x(2)), \quad \ldots.$$

Set $\phi(t, x_0)$ to be the solution $x(t)$ with initial condition $x(0) = x_0$. We make this notation to keep track of the initial condition. Then

$$\frac{\partial \phi(t,x_0)}{\partial t} \ = \ f(t, \phi(t,x_0)).$$

Notice that

$$p(x_0) \ = \ \phi(1, x_0).$$

In the particular case of the logistic equation with periodic harvesting, we can compute formulas for the Poincare map. By contrast, for most equations this cannot be done so explicitly.

Remark. The solution $\phi(t,x_0)$ is periodic with period 1 if and only if $p(x_0) = x_0$.

Step 1. Suppose $x_0 > 1$. Set $x_1 = p(x_0)$, $x_2 = p(x_1)$, etc.. Then by the inequality (**), x_j is bounded above by a sequence which converges to 1, and hence for j sufficiently large we have $x_j < x_0$. However, this means that for some $k < j$ we have $p(x_k) = x_{k+1} < x_k$.

Step 2. Suppose that there exists a point y with $p(y) > y$. Then assuming that p is continuous, there exists some point x between x_k and y with $p(x) = x$. This is seen by applying the intermediate value theorem to the function $p(x) - x$. Since this function is positive at y and negative at x_k, it must assume the value zero somewhere in between.

Step 3. If p has no fixed points, that is if there is no value of x with $p(x) = x$, then the population eventually becomes extinct.

Proof. If there is no fixed point then by Step 2, we see that for every value of x we have $p(x) < x$. Start with $x_0 > 0$ and set

$$x_1 = p(x_0), \qquad x_2 = p(x_1), \ldots .$$

Then x_j is decreasing. If the sequence x_j is non-negative then it converges to some value $x \geq 0$. However, assuming p is continuous this means that $p(x) = p(\lim x_j) = \lim p(x_j) = \lim x_{j+1} = x$. Hence x is a fixed point of p which is a contradiction. Hence eventually x_j must be negative.

Strategy. We will show that there are at most two values of x_0 which yield periodic solutions. The solution $\phi(t, x_0)$ is periodic if and only if $p(x_0) = x_0$.

Main formula: Following page 13,

$$\begin{aligned} p'(x_0) &= \exp\left(\int_0^1 (1 - 2\phi(t, x_0))\, dt\right) \\ p''(x_0) &= -2a^2\, p'(x_0) \int_0^1 \exp\left(\int_0^s (1 - 2\phi(t, x_0))\, dt\right) ds. \end{aligned}$$

We see that $p'(x_0) > 0$ and $p''(x_0) < 0$, so p is an increasing function which is concave down. Hence it intersects that diagonal $p(x) = x$ in at most 2 points.

Now as h increases, the value of $f(t, x)$ decreases. Hence the value of the solution $\phi(t, x_0)$ decreases as h increases and so does the poincare map $p(x_0)$. There is a unique point h_* for which there is one fixed point of p, and for $h > h_*$ there are no fixed points and the population becomes extinct.