Math 130A. Ordinary Differential Equations

Lecture 5: Periodic harvesting.

Last time: Logistic Population Model with periodic harvesting:

$$(*) \qquad x' = f(t,x) = ax(1-x) - h(1+\sin(2\pi t)).$$

Set $\phi(t,x_0)$ to be the solution $x(t)$ with initial condition $x(0) = x_0$, and define the Poincare map by

$$p(x_0) = \phi(1, x_0).$$

Then if p has no fixed points, the population becomes extinct. This happens for $h > h_*$. Since $p''(x_0) > 0$, we deduce that p has at most two fixed points. Each fixed point corresponds to a periodic solution. A periodic solution with initial population x_0 ensures that the population cannot become extinct provided $x \geq x_0$.

Main formula: Following page 13,

$$\begin{aligned} p'(x_0) &= \exp\left(\int_0^1 a\,(1-2\phi(t,x_0))\,dt\right) \\ p''(x_0) &= -2a\;p'(x_0)\int_0^1 \exp\left(\int_0^s a\,(1-2\phi(t,x_0))\,dt\right)ds. \end{aligned}$$

Proof. We follow pages 13 and 14. Since

$$(1) \qquad \frac{\partial \phi}{\partial t}(t,x_0) = f(t,\phi(t,x_0)),$$

and $\phi(0,x_0) = x_0$, we integrate to get

$$\phi(t,x_0) = x_0 + \int_0^t f(s,\phi(s,x_0))\,ds.$$

Differentiating with respect to x_0, we get

$$(2) \qquad \frac{\partial \phi}{\partial x_0}(t,x_0) = 1 + \int_0^t \frac{\partial f}{\partial x_0}(s,\phi(s,x_0)) \cdot \frac{\partial \phi}{\partial x_0}(s,x_0)\,ds.$$

Now let

$$z(t) = \frac{\partial \phi}{\partial x_0}(t,x_0).$$

Note that

$$z(0) = \frac{\partial \phi}{\partial x_0}(0,x_0) = 1.$$

Differentiating either (1) with respect to x_0, or (2) with respect to t, we get

$$z'(t) = \frac{\partial^2 \phi}{\partial x_0 \partial t} = \frac{\partial f}{\partial x_0}(t,\phi(t,x_0)) \cdot \frac{\partial \phi}{\partial x_0}(t,x_0) = \frac{\partial f}{\partial x_0}(t,\phi(t,x_0)) \cdot z(t).$$

This gives a differential equation for $z(t)$. Note that the solution to

$$z'(t) \;=\; F(t)\cdot z(t), \qquad\qquad z(0)=1,$$

is

$$z(t) \;=\; \exp\int_0^t F(s)\,ds.$$

In our case, this gives

$$z(t) \;=\; \exp\int_0^t \frac{\partial f}{\partial x_0}(s,\phi(s,x_0))\,ds.$$

Hence

$$p'(x_0) \;=\; \frac{\partial \phi}{\partial x_0}(1,x_0) \;=\; z(1) \;=\; \exp\int_0^1 \frac{\partial f}{\partial x_0}(s,\phi(s,x_0))\,ds.$$

From this we see that p' is positive. Differentiating again we get

$$\begin{aligned} p''(x_0) \;&=\; p'(x_0)\,\frac{d}{dx_0}\int_0^1 \frac{\partial f}{\partial x_0}(s,\phi(s,x_0))\,ds \\ &=\; p'(x_0)\,\int_0^1 \frac{\partial^2 f}{\partial x_0 \partial x_0}(s,\phi(s,x_0))\frac{\partial \phi}{\partial x_0}(s,x_0)\,ds \\ &=\; p'(x_0)\,\int_0^1 \frac{\partial^2 f}{\partial x_0 \partial x_0}(s,\phi(s,x_0))z(s)\,ds \\ &=\; p'(x_0)\,\int_0^1 \frac{\partial^2 f}{\partial x_0 \partial x_0}(s,\phi(s,x_0))\exp\int_0^s \frac{\partial f}{\partial x_0}(u,\phi(u,x_0))\,du\,ds \end{aligned}$$

Computing from the definition of f:

$$\begin{aligned} \frac{\partial f}{\partial x_0}(t,x_0) \;&=\; a(1-2x_0), \\ \frac{\partial^2 f}{\partial x_0^2}(t,x_0) \;&=\; -2a. \end{aligned}$$

This gives the result.