Math 130A. Ordinary Differential Equations

Lecture 6: Linear Systems.

Systems of ODEs. Consider the second order differential equation

$$(*) \qquad x'' = f(t, x, x').$$

Examples. Newton's Law

$$mx'' = f(x),$$

Forced harmonic oscillator:

$$mx'' + bx' + kx = f(t),$$

where m is the mass, $b \geq 0$ is the damping constant, $k > 0$ is the spring constant, and $f(t)$ is the external force.

In general (*) can be rewritten as a first order **system**:

$$\begin{aligned} x' &= y \\ y' &= f(t, x, y) \end{aligned}$$

Examples. We can draw direction fields for

$$\begin{aligned} x' &= y \\ y' &= -x \end{aligned}$$

$$\begin{aligned} x' &= y \\ y' &= x. \end{aligned}$$

We use unit vectors to simplify the picture.

A planar linear system is a system of ODEs of the form

$$(*) \qquad \begin{aligned} x' &= ax + by \\ y' &= cx + dy, \end{aligned} \qquad i.e. \qquad \begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Set

$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \qquad X = \begin{pmatrix} x \\ y \end{pmatrix}.$$

The **equilibrium points** are those $\begin{pmatrix} x \\ y \end{pmatrix}$ with $\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. This occurs when

$$AX = 0,$$

that is, for such an $X = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}$, we get a constant solution to the equation (*) given by

$$\begin{aligned} x(t) &= x_0, \\ y(t) &= y_0. \end{aligned}$$

Review of linear algebra:

The following are equivalent.

1. The only solution X to the equation $AX = 0$ is the solution $X = 0$. (In other words the null space of A contains only the zero vector.)

2. The determinant $\det A = ad - bc$ *is non-zero.*

3. There exists an inverse matrix A^{-1} given by

$$A^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}.$$

4. For every vector Y in $\mathbb{R}^2$, there is a unique solution to the equation $AX = Y$.

5. The columns of the matrix A span $\mathbb{R}^2$.

6. The columns of A are linearly independent.

The following are equivalent. *assuming that A is not the zero matrix $\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.*

1. There exists a non-zero solution to the equation $AX = 0$.

2. The null space of A forms a straight line through the origin spanned by the vectors $\begin{pmatrix} b \\ -a \end{pmatrix}$, $\begin{pmatrix} d \\ -c \end{pmatrix}$ (which are linearly dependent)

3. A is singular. (It does not have an inverse.)

4. There exists a vector Y in $\mathbb{R}^2$ such that the equation $AX = Y$ has no solution.

5. The columns of A span a line.

6. The columns of A are linearly dependent.

Algebraic Solution of the Equation (*)

Theorem. *There exists a change of basis matrix P such that*

$$A = PGP^{-1}$$

where G is either the diagonal matrix $\begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$ or the matrix $\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$.

Using this we can solve the equation

$$X' = AX.$$

Indeed, it becomes

$$X' = PGP^{-1}X.$$

However, multiplying by P^{-1} gives

$$P^{-1}X' \;=\; GP^{-1}X.$$

However,

$$P^{-1}\;X' \;=\; \begin{pmatrix} jx' + ky' \\ mx' + \ell y' \end{pmatrix} \;=\; \begin{pmatrix} (jx + ky)' \\ (mx + \ell y)' \end{pmatrix} \;=\; (P^{-1}X)',$$

so setting $Y = P^{-1}X$, we have

$$Y' \;=\; GY.$$

Setting $Y = \begin{pmatrix} u \\ v \end{pmatrix}$, in the first case we get

$$\begin{aligned} u' &= \lambda u \\ v' &= \mu v \end{aligned}$$

and so

$$\begin{aligned} u(t) &= u(0)e^{\lambda t} \\ v(t) &= v(0)e^{\mu t}. \end{aligned}$$

We then recover X by the formula $X = PY$. We find that the solution is of the form

$$\begin{aligned} x(t) &= \alpha e^{\lambda t} \;+\; \beta e^{\mu t} \\ y(t) &= \gamma e^{\lambda t} \;+\; \delta e^{\mu t}, \end{aligned}$$

for constants $\alpha, \beta, \gamma, \delta$. We consider the second case next time.