Lecture 8: Linear Algebra.

Linear Algebra. If $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then $A = PGP^{-1}$ where either $G = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ with λ_1, λ_2 real or $G = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ with λ real or $G = \begin{pmatrix} \alpha + i\beta & 1 \\ 0 & \alpha - i\beta \end{pmatrix}$ and P is complex. In the third case, we can get $A = QHQ^{-1}$ with Q real and $H = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$.

Proof. We consider the characteristic polynomial. If $T = a + d$ is the trace of A and $D = ad - bc$ is the determinant of A then

$$\det(A - \lambda I) \;=\; \lambda^2 \;-\; T\lambda \;+\; D \;=\; (\lambda - T/2)^2 \;+\; D \;-\; T^2/4.$$

Hence $\det(A - \lambda I) = 0$ precisely when

$$\lambda \;=\; \frac{T}{2} \;\pm\; \frac{\sqrt{T^2 - 4D}}{2}.$$

We have three cases.

(1) $T^2 - 4D > 0$. Then A has two real distinct roots λ_1, λ_2. In this case, A is diagonalizable. Indeed, we get eigenvectors V_1 and V_2 with

$$AV_1 = \lambda_1 V_1, \qquad\qquad AV_2 = \lambda_2 V_2.$$

If $V_1 = cV_2$ then

$$\lambda_1 V_1 \;=\; AV_1 \;=\; A(cV_2) \;=\; cAV_2 \;=\; c\lambda_2 V_2 \;=\; \lambda_2 V_1.$$

Hence $\lambda_1 = \lambda_2$. So V_1 and V_2 are linearly independent and if $P = [V_1\,,\, V_2]$ then

$$AP \;=\; A[V_1\,,\, V_2] \;=\; [\lambda_1 V_1\,,\, \lambda_2 V_2] \;=\; [V_1\,,\, V_2]\begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} \;=\; PG.$$

(2) $T^2 - 4D = 0$. In this case A has one repeated eigenvalue. Since $A - \lambda I$ is not invertible, this means that there exists an eigenvector V with eigenvalue λ. Let W be any vector linearly independent from V. Then write

$$W \;=\; \mu W \;+\; \nu V.$$

If $\nu = 0$ then W is an eigenvector for A with eigenvalue ν and hence $\mu = \lambda$ and $A = \lambda I = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$. Indeed, for any vector $\alpha V + \beta W$ we have

$$A(\alpha V + \beta W) \;=\; \alpha AV \;+\; \beta AW \;=\; \alpha\lambda V \;+\; \beta\lambda W \;=\; \lambda(\alpha V + \beta W).$$

We see that this gives a special case of the first sort of matrix rather than the second sort. If $\nu \neq 0$ then by rescaling W we can assume $\nu = 1$, so

$$AW \;=\; \mu W \;+\; V.$$

We claim that μ is an eigenvalue of A. Indeed, writing the linear map $U \to AU$ with respect to the basis V, W, we have matrix

$$A \;\sim\; \begin{pmatrix} \lambda & 1 \\ 0 & \mu \end{pmatrix}.$$

But then

$$A - \mu I \;\sim\; \begin{pmatrix} \lambda - \mu & 1 \\ 0 & 0 \end{pmatrix}.$$

This matrix has eigenvector $\begin{pmatrix} 1 \\ \mu - \lambda \end{pmatrix}$ which is expressed with respect to the basis V, W. Hence it corresponds to the vector

$$U \;=\; V \;+\; (\mu - \lambda)W.$$

We check that

$$AU \;=\; A(V + (\mu - \lambda)W) \;=\; AV \;+\; (\mu - \lambda)AW \;=\; \lambda V \;+\; (\mu - \lambda)\mu W \;+\; (\mu - \lambda)V$$
$$=\; \mu(V \;+\; (\mu - \lambda)W) \;=\; \mu AU.$$

(3) $T^2 - 4D < 0$. In this case A has complex roots $\alpha + i\beta$ and $\alpha - i\beta$. We get a complex eigenfunction $V_1 + iV_2$ corresponding to $\alpha + i\beta$. Then

$$A(V_1 + iV_2) = (\alpha + i\beta)(V_1 + iV_2) \;=\; (\alpha V_1 - \beta V_2) \;+\; i(\beta V_1 + \alpha V_2).$$

Hence

$$AV_1 \;=\; (\alpha V_1 - \beta V_2), \qquad AV_2 \;=\; (\beta V_1 + \alpha V_2).$$

If we can show that V_1 and V_2 form a basis, then with respect to this basis, A has matrix $\begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix}$. We also remark that $A(V_1 - iV_2) \;=\; (\alpha - i\beta)(V_1 - iV_2)$. To see that V_1 and V_2 are linearly independent, first note that $V_2 \neq 0$ or else the eigenvector is real and hence the eigenvalue must be real. Suppose that $V_1 = cV_2$. Then $V_1 + iV_2 = (c + i)V_2$ and hence

$$(c+i)AV_2 \;=\; A((c+i)V_2) \;=\; A(V_1+iV_2) \;=\; (\alpha+i\beta)(V_1+iV_2) \;=\; (\alpha+i\beta)(c+i)V_2.$$

But then dividing by $c + i$ we get $AV_2 = (\alpha + i\beta)V_2$ and so $\beta = 0$, that is the characteristic equation has a repeated real root which is a contradiction.