MATH 130A. ORDINARY DIFFERENTIAL EQUATIONS

Lecture 9: Phase portraits.

Example 1. Sketch the phase portrait for the system

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Describe the behavior as $t \to \infty$.

Solution. We solve for y and then x to get

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} (x(0) + y(0)t)e^{\lambda t} \\ y(0)e^{\lambda t} \end{pmatrix}.$$

From this we see that if $\lambda > 0$ then $x \to \pm\infty$ and $y \to \pm\infty$ as $t \to \infty$, where the sign agrees with that of $y(0)$, while if $\lambda < 0$ then $x \to 0$ and $y \to 0$ as $t \to \infty$. If $\lambda = 0$ then y is constant and $x \to \pm\infty$ as $t \to \infty$ where the sign agrees with that of $y(0)$.

$y(t)$ always has the same sign as $y(0)$, and

$$x(t) = \frac{y(t)}{y(0)} \left(x(0) + \frac{y(0)}{\lambda} \log \frac{y(t)}{y(0)} \right).$$

The solution curve through $(x(0), y(0)) = (x_0, \pm 1)$ is given by

$$x(t) = y(t) \left(x_0 + \frac{\log |y(t)|}{\lambda} \right) = \pm \left(x_0 |y(t)| + \frac{|y(t)| \log |y(t)|}{\lambda} \right).$$

Example 2. Sketch the phase portrait for the system

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \alpha & \beta \\ -\beta & \alpha \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}.$$

Solution. Characteristic polynomial is $(\alpha - \lambda)^2 + \beta^2 = \lambda^2 - 2\alpha\lambda + (\alpha^2 + \beta^2)$. The roots are $\alpha \pm i\beta$. The complex eigenvector for the root $\alpha + i\beta$ is given by solving

$$\begin{pmatrix} -i\beta & \beta \\ -\beta & -i\beta \end{pmatrix} V = 0.$$

We get

$$V = \begin{pmatrix} -i \\ 1 \end{pmatrix}.$$

Theorem. . *If V_0 is an eigenvector of A with eigenvalue λ, then $e^{t\lambda}V_0$ solves $X' = AX$.*

Indeed,

$$\left(e^{t\lambda}V_0\right)' \;=\; \lambda e^{t\lambda}\lambda V_0 \;=\; e^{t\lambda}AV_0 \;=\; A(e^{t\lambda}V_0).$$

This gives the complex solution

$$e^{t(\alpha+i\beta)}\begin{pmatrix}1\\ i\end{pmatrix} \;=\; e^{\alpha t}\begin{pmatrix}\cos\beta t + i\sin\beta t\\ -\sin\beta t \;+\; i\cos\beta t\end{pmatrix}.$$

Taking real and imaginary parts gives the two real solutions

$$e^{\alpha t}\begin{pmatrix}\cos\beta t\\ -\sin\beta t\end{pmatrix}, \qquad\qquad e^{\alpha t}\begin{pmatrix}\sin\beta t\\ \cos\beta t\end{pmatrix}.$$

Any linear combination of these forms a solution,

$$\begin{pmatrix}x(t)\\ y(t)\end{pmatrix} \;=\; c_0\; e^{\alpha t}\begin{pmatrix}\cos\beta t\\ -\sin\beta t\end{pmatrix} \;+\; c_1\; e^{\alpha t}\begin{pmatrix}\sin\beta t\\ \cos\beta t\end{pmatrix}.$$

At $t = 0$ this is

$$c_0\;\begin{pmatrix}1\\ 0\end{pmatrix} \;+\; c_1\;\begin{pmatrix}0\\ 1\end{pmatrix}.$$

We can get the general solution this way since we can get any initial condition $\begin{pmatrix}x(0)\\ y(0)\end{pmatrix}$ by taking

$$\begin{pmatrix}x(t)\\ y(t)\end{pmatrix} \;=\; x(0)\; e^{\alpha t}\begin{pmatrix}\cos\beta t\\ -\sin\beta t\end{pmatrix} \;+\; y(0)\; e^{\alpha t}\begin{pmatrix}\sin\beta t\\ \cos\beta t\end{pmatrix}.$$

We see that if $\alpha = 0$ then $\begin{pmatrix}x(t)\\ y(t)\end{pmatrix}$ travels in a circle around the origin. If $\alpha > 0$ then it spirals inwards as $t \to \infty$ ....

We can calculate the direction of the spiral by noticing that

$$\begin{pmatrix}\cos\beta t\\ -\sin\beta t\end{pmatrix}, \begin{pmatrix}\sin\beta t\\ \cos\beta t\end{pmatrix}$$

are orthogonal unit vectors which rotate around the origin with speed β radians per unit time in the clockwise direction. (If $\beta < 0$ then they rotate counterclockwise with speed $|\beta|$.) Hence the vector

$$x(0)\;\begin{pmatrix}\cos\beta t\\ -\sin\beta t\end{pmatrix} \;+\; y(0)\;\begin{pmatrix}\sin\beta t\\ \cos\beta t\end{pmatrix}.$$

also rotates around the origin with speed β in the clockwise direction. Multiplying by $e^{t\alpha}$ makes this vector spiral outwards to infinity as $t \to \infty$ if $\alpha > 0$, and it spirals inwards towards zero as $t \to \infty$ if $\alpha < 0$. If $\alpha = 0$ then it moves in a circle. These cases are called a spiral source, spiral sink, and center respectively.