2a) NOTE: SAYING THAT "THIS FOLLOWS FROM HOMEWORK PROBLEMS 2.22 AND 2.23" DOES NOT CONSTITUTE A PROOF AND WILL THUS BE WORTH ZERO POINTS.

THIS PROBLEM CONSISTS OF TWO PARTS.

i) SHOWING THAT SOME SET OF OPEN BALLS IN $\mathbb{R}^2$ CONSTITUTES A COUNTABLE BASE;

ii) SHOWING THAT ANY OPEN $E \subseteq \mathbb{R}^2$ CAN BE WRITTEN AS A UNION OF BALLS IN THE BASE.

PART (i) IS WORTH 6 POINTS. SUITABLE CANDIDATES FOR THE BASE LOOK A LOT LIKE

$$V = \{B_q(p) : p \in \mathbb{Q}^2,\ q \in \mathbb{Q}\}$$

COUNTABILITY - 2 PTS
BASE - 4 PTS

PART (ii) IS WORTH 6 POINTS. ONE MUST SHOW

$$E = \bigcup_{\substack{p,q \\ B_q(p) \subseteq E \cap V}} B_q(p)$$

ONE INCLUSION IS OBVIOUS (2 PTS), THE OTHER SLIGHTLY LESS OBVIOUS (4 PTS).

b) THE EASIEST APPROACH MAY BE TO

i) SHOW EACH OPEN BALL IS A COUNTABLE UNION OF CLOSED BALLS (8 PT);

ii) APPEAL TO (a) (2 PT);

iii) NOTE THAT THE COUNTABLE UNION OF COUNTABLE SETS IS COUNTABLE (2 PT);

PART i FOLLOWS FROM NOTING THAT

$$B_r(x) = \bigcup_{n=1}^{\infty} \overline{B_{r(1-\frac{1}{n})}(x)}.$$

4. EACH PART IS WORTH 6 POINTS.

a) NONE EXISTS. THIS FOLLOWS FROM THE FACT THAT $[0,1]$ IS COMPACT (2 PTS: H-B), BUT $\mathbb{R}$ IS NOT. (THE FACT THAT $\mathbb{R}$ IS NOT COMPACT CAN BE CITED WITHOUT PROOF).

b) NONE EXISTS. IF f^{-1} IS CONTINUOUS, THEN B BEING CONNECTED IMPLIES $f^{-1}(B)$ IS AS WELL, WHICH IS NOT THE CASE (THE FACT THAT A IS SEPARATED IS OBVIOUS ENOUGH TO NOT REQUIRE A PROOF). APPLYING THIS ARGUMENT TO f RATHER THAN f^{-1} COSTS 2 PTS.

c) THERE ARE SEVERAL EXAMPLES THAT DO THE JOB; ONE EXAMPLE IS

$$f: A \longrightarrow B, \quad f(x) = \tan\left(\frac{\pi}{2}x\right).$$

MINOR ERRORS IN THE CONSTRUCTION OF f (HAVING ITS DOMAIN IMPROPERLY DEFINED, FOR EXAMPLE) WILL COST 1 POINT.

d) NONE EXISTS. APPEALS MAY BE MADE EITHER TO THE INTERMEDIATE VALUE THEOREM, OR A SIMILAR ARGUMENT TO (b) MAY BE MADE ON f^{-1} RESTRICTED TO $B \setminus \{0\} = (0, \infty)$.

5. EACH PART IS WORTH 6 POINTS. CHOOSING THE INCORRECT OPTION IS WORTH 0 OF 6.

a) FALSE.

LET $A = [0,1]$, $B = [1,2]$. EACH IS PERFECT (CAN CITE W/OUT PROOF). YET $A \cap B = \{1\}$. $A \cap B$ IS NOT PERFECT, SINCE ANY NONEMPTY PERFECT SET MUST BE UNCOUNTABLE (THM 2.43).

b) FALSE. SEE BELOW (*).

LET $X = \mathbb{R}$, $f(x) = \cos^2 x$, $x_n = \frac{n\pi}{2}$. THEN

$$f(x_n) = \begin{cases} 1 & n \text{ EVEN} \\ 0 & n \text{ ODD} \end{cases}$$

AND THUS $f(x_n)$ IS DIVERGENT.

(THE DIVERGENCE OF $f(x_n)$ IN THIS CASE IS OBVIOUS).

c) TRUE. IF $p_n \longrightarrow p$, THEN $\forall\ \varepsilon > 0$, $\exists\ N_\varepsilon \in \mathbb{N}$ S.T.
$d(p_n, p) < \varepsilon\ \ \forall\ n > N_\varepsilon$.

FIX $\varepsilon > 0$. THEN IF $n > N_\varepsilon$, $d(m_n, p) < \varepsilon$ BY DEFINITION OF m_n. (MORE SHOULD BE SAID HERE; FAILURE TO PROVIDE DETAILS WILL COST 1-2 POINTS).

THUS $m_n \longrightarrow p$.

d) FALSE. ANY NUMBER OF EXAMPLES WILL SUFFICE; FOR ONE,

$$x_n := n \quad , \; y_n := -n.$$

GIVES $x_n + y_n = 0 \longrightarrow 0$,
$x_n - y_n = 2n$ DIVERGENT.

(*) IF, IN PART (b), YOU TAKE

"f IS CONTINUOUS IFF $\forall\ x_n \to x$, $f(x_n) \to f(x)$"

TO IMPLY

"f IS CONTINUOUS $\Rightarrow \forall \{x_n\}$ DIVERGENT, $\{f(x_n)\}$ IS DIVERGENT",

-4 PTS.

6. EACH PART IS WORTH 8 POINTS.

<u>NOTE</u>: THE SET G IS REALLY THE FOLLOWING SET IN DISGUISE:

$$\{2\times 2 \text{ REAL MATRICES } A : \det A \neq 0\}.$$

a) IT SUFFICES TO SHOW THAT ANY $x=(a,b,c,d)\in\mathbb{R}^4$ WITH $ad-bc=0$ IS A LIMIT POINT OF G; I.E., EVERY NEIGHBORHOOD OF X CONTAINS A MEMBER OF G. (2 PTS).

SHOWING THIS FACT IS WORTH 6 PTS. IT MAY BE BEST TO LOOK AT THREE CASES

I: $x=(0,0,0,0)$

II: $(a=0$ OR $d=0)$ AND $(b=0$ OR $c=0)$

III: a,b,c,d ARE NONZERO AND $ad=bc$.

b) THE EASIEST APPROACH IS TO USE THE HINT; SINCE $f:\mathbb{R}^4\longrightarrow\mathbb{R}$ DEFINED BY

$$f(a,b,c,d)=ad-bc$$

IS CONTINUOUS (4 PTS), THE SET $f^{-1}(\{0\})$ IS CLOSED, SO ITS COMPLEMENT, G, IS OPEN. (4 PTS)

c) BY HEINE-BOREL, $\mathbb{R}^4\setminus G$ IS COMPACT IFF IT IS CLOSED AND BOUNDED; HOWEVER, $\forall\, N\in\mathbb{N}$,

$$(N,0,0,0)\in\mathbb{R}^4\setminus G.$$

HENCE, $\mathbb{R}^4\setminus G$ IS NOT COMPACT.