

Problem 1. (10%). Suppose that S and T are sets, and $f : S \rightarrow T$ is a function (not necessarily onto). For each of the following statements, either prove that it holds for every function f , or give a function f for which it is false.

- (a) $f^{-1}(X \cap Y) = f^{-1}(X) \cap f^{-1}(Y)$ for all $X, Y \subset T$.
 (b) $f(U \cap V) = f(U) \cap f(V)$ for all $U, V \subset S$.

Problem 2. (10%). Let S be a subset of \mathbb{R}^n .

- (a). Define what it means for $p \in S$ to be an accumulation point of S .
 (b). Writing S' for the set of accumulation points of S , show that S' is closed.

Problem 3. (10%). Determine whether the following series converge, naming any test that you use.

(a). $\sum_{n=2}^{\infty} \frac{1}{n \log n}$. (b). $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \log n}$.

Problem 4. (12%). Let (S, d) be a metric space and let $f : S \rightarrow \mathbb{R}$ and $g : S \rightarrow \mathbb{R}$ be functions from S into \mathbb{R} .

- (a). Define what it means for the function f to be continuous at $p \in S$.
 (b). Show that if f and g are both continuous at p then the function $f - g$ is also continuous at p .
 (c). Suppose f and g are both continuous on S , and set

$$A = \{p \in S : f(p) = g(p)\}.$$

What can you say about the set A ? Explain your answer. (Hint: Use part (b).)

Problem 5 (10%). Set

$$f(x) = \sum_{n=2}^{\infty} \sqrt{n} x^n.$$

- (a). For which real values x does the series $f(x)$ converge? Justify your answer.
 (b). Choose real numbers a and b with $a < b$ so that f defines a homeomorphism from $[a, b]$ onto its image. Explain why the values you chose work.

Problem 6 (12%). In the following cases, does there exist a continuous function $f : S \rightarrow T$ with $f(S) = T$? Justify your answers.

- (a) $S = [0, 1], \quad T = [1, \infty).$
 (b) $S = [0, 1] \cup [2, 3], \quad T = [0, 1].$
 (c) $S = [0, 1], \quad T = [0, 1] \cup [2, 3].$

Problem 7. (10%). Suppose S and T are subsets of \mathbb{R} . Define

$$S \times T = \{(s, t) : s \in S, t \in T\} \subseteq \mathbb{R}^2, \quad S - T = \{s - t : s \in S, t \in T\} \subseteq \mathbb{R}.$$

Decide whether the following statements are true or false and give an explanation or counterexample.

- (a). If S and T are closed in \mathbb{R} , then $S \times T$ is closed in \mathbb{R}^2 .
 (b). If S and T are compact in \mathbb{R} then $S \times T$ is compact in \mathbb{R}^2 ?
 (c). If S and T are compact in \mathbb{R} then $S - T$ is compact in \mathbb{R} .
 (d). If S and T are closed in \mathbb{R} then $S - T$ is closed in \mathbb{R} .

Problem 8 (11%). Let (S, d_S) and (T, d_T) be metric spaces. Let $f : S \rightarrow T$ be a function.

- (a). Define what it means for f to be uniformly continuous.
 (b). Suppose that $f : (0, 1] \rightarrow \mathbb{R}$ is uniformly continuous. Show that

$$\lim_{x \rightarrow 0} f(x)$$

exists. (Hint: Use sequences.)

- (c). Show that if $f : \mathbb{R} \rightarrow \mathbb{R}$ is uniformly continuous, then there exists a constant C such that

$$|f(x)| \leq C(1 + |x|) \quad \text{for all } x \in \mathbb{R}.$$

Problem 9. (15%). Decide whether the following statements are true or false. If the statement is true then give a brief explanation. If the statement is false then give a counterexample.

- (a). An uncountable subset U of \mathbb{R} must contain an irrational number.
 (b). There exists an uncountable collection of non-empty pairwise disjoint open subsets of \mathbb{R}^2 .

- (c). If (x_n) is a real sequence converging to x and $x_n < 1$ for all n then $x < 1$.
- (d). If $F_0 \supseteq F_1 \supseteq F_2 \supseteq \dots$ is a sequence of non-empty closed subsets of the set $[0, 1] \times [0, 1] \subset \mathbb{R}^2$, then $\bigcap_{n=1}^{\infty} F_j$ is non-empty.
- (e). Suppose (S, d_S) is a metric space and $A \subset X \subset S$ are subsets. If A is open in the subspace X , then it is open in S .