Math 140A. Foundations of Analysis.
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**Lecture 16. Compactness.** Here are the solutions you came up with in class:

**Old midterm problem (2008)**: Decide whether the following statements are true or false.

(a) A countable intersection of open subsets of $\mathbb{R}$ is open.

False. For example in $\mathbb{R}^k$,

$$\bigcap_{k=1}^{\infty} N_{1/k}(0) \ = \ \{0\}.$$

In $\mathbb{R}^1$, the neighborhoods (or balls) are just intervals. We have

$$\bigcap_{k=1}^{\infty} (-1/k, 1/k) \ = \ \{0\}.$$

(b) If $F$ is a function and $A$, $B$ are subsets of the domain of $F$, then $F(A \cup B) = F(A) \cup F(B)$.

True. Suppose $y \in F(A \cup B)$. Then $y = f(x)$ where $x \in A \cup B$. Then either $x \in A$ or $x \in B$. In the first case, $y \in F(A) \subset F(A) \cup F(B)$. In the second case, $y \in F(B) \subset F(A) \cup F(B)$, so we see that in either case, $y \in F(A) \cup F(B)$. Hence $F(A \cup B) \subset F(A) \cup F(B)$.

Conversely, suppose $y \in F(A) \cup F(B)$. Then either $y \in F(A)$ or $y \in F(B)$. In the first case, $y = F(x)$ with $x \in A \subset A \cup B$ and so $y \in F(A \cup B)$. In the second case, $y = F(x)$ with $x \in B \subset A \cup B$ and so again, $y \in F(A \cup B)$. Hence in either case, $y \in F(A \cup B)$ and so $F(A) \cup F(B) \subset F(A \cup B)$.

Since we obtained both inclusions, we conclude that $F(A \cup B) = F(A) \cup F(B)$.

(c) If $A$ is a countable subset of an uncountable set $B$, then $B - A$ is uncountable.

True. Suppose to get a contradiction that $B - A$ is countable. Then $A = B \cup (A - B)$ is a union of two countable sets and hence countable, which is a contradiction. Hence $B - A$ must be uncountable.

(d) If $S$ is an infinite subset of $\mathbb{R}$ then $S$ has an accumulation point in $\mathbb{R}$.

False. For example, the integers have no accumulation point in $\mathbb{R}$.

(e) If $S$ is an uncountable subset of $\mathbb{R}$ then $S$ has an accumulation point in $\mathbb{R}$.

True. (The class combined their ideas to give two essentially different proofs of this fact.) The results we need are

**Theorem 2.37**. If $E$ is an infinite subset of a compact set $K$, then $E$ has a limit point in $K$.

**Theorem 2.40**. Every $k$-cell is compact.

First proof of (e). Notice that $[-n,n]\cap S$ must be infinite (in fact uncountable) for some $n$. Otherwise $S=\cup_{n=1}^{\infty}([-n,n]\cap S)$ is a countable union of finite sets and hence countable. However, since $[-n,n]\cap S$ is infinite, and $[-n,n]$ is compact, we see that $S$ has a limit point in $[-n,n]$.

Second proof of (e). (This proof is stronger because it shows that $S$ has a limit point which lies in $S$.) Suppose that no point of $S$ is a limit point of $S$. Then for each $p\in S$ there exists $r_p>0$ such that $p$ is the only point of $S\cap N_{r_p}(p)$. If $p$ and $q$ are points of $S$, then

$$(*) \qquad N_{r_p/2}(p)\cap N_{r_q/2}(q) \;=\; \emptyset.$$

To see this, suppose without loss of generality that $r_p\geq r_q$, and suppose that there exists some point $x$ in $N_{r_p/2}(p)\cap N_{r_q/2}(q)$. Then

$$|p-q| \;\leq\; |p-x| \;+\; |x-q| \;\leq\; \frac{r_p}{2} \;+\; \frac{r_q}{2} \;\leq\; r_p.$$

Hence $q\in N_{r_p}(p)$ which is a contradiction. Hence (*) holds. Now since the rationals are dense in $\mathbb{R}$, for each $p\in S$ we can choose a rational $w_p\in N_{r_p/2}(p)$. Clearly if $p\neq q$ then $w_p\neq w_q$ by (*). But the rationals are countable and so this establishes a 1-1 correspondence between points $p$ in $S$ and the points $w_p$ which form a countable set. Hence $S$ is countable. Hence if $S$ is uncountable then $S$ must have a limit point in $S$.