Math 140A. Foundations of Analysis.
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Lecture 2: The real number system.

Last time: Recall that an ordered set is a set S with a relation $<$ such that for every two elements $x, y \in S$, only one of the following holds: $x < y$ or $x = y$ or $y < x$, and $<$ is transitive.

If S is an ordered set and E is a subset of S, then the **supremum** E denoted by $\sup E$, is the least upper bound of E in S if it exists. Clearly if the supremum of E exists then it is unique.
Remark. Unless E contains its own supremum, then the supremum of a set E depends on the set S. For example, let E be the set of negative rationals. Find the supremum of E in S where S denotes the following sets with the usual ordering:

$$(a) \qquad S = E \cup \{1\}. \qquad\qquad (b) \qquad S = E \cup \{0,1\}.$$

An ordered set S is said to have the **least upper bound property** if whenever $E \subset S$ is non-empty and bounded above, then the supremum of E exists in S.

An ordered set S is said to have the **greatest lower bound property** if whenever $E \subset S$ is non-empty and bounded below, then the infimum of E exists in S.

We saw that the rationals do not satisfy the least upper bound property.

Theorem 1.11. *Suppose S is an ordered set with the least-upper-bound property. Then S also has the greatest lower bound property.*

Proof. Supppose that $B \subset S$ is non-empty and bounded below. Let

$$L = \{\beta : \beta \text{ is a lower bound for } B\}.$$

If $\beta \in L$ then $\beta \leq x$ for every $x \in B$. Hence every $x \in B$ is an upper bound for L. In particular L is bounded above. Then $\alpha = \sup L$ exists in S. It is clear that α is a lower bound for B. Indeed, since every $x \in B$ is an upper bound for L, we see that $\beta = \sup L \leq x$. Finally we must show that α is the least upper bound for B. But this is obvious since it is an upper bound for L.

Now we are going to state the algebraic properties of the real numbers. Algebra is simple and orderly and unambiguous!

We will follow the book to define a field and an ordered field as in 1.12-1.18. The point is the following:

Theorem 1.19. *There exists an ordered field $\mathbb{R}$ which has the least upper bound property. Moreover, $\mathbb{R}$ contains $\mathbb{Q}$ as a subfield.*

In fact $\mathbb{R}$ is the "unique" ordered field with the least upper bound property.