

**Lecture 3: The real number system.**

This lecture we mostly followed the book to prove Theorems 1.20 and 1.21.

We just comment on the proof of Theorem 1.20 (b). We first choose  $n$  large enough so that  $1/n < y - x$ . The reason for doing this is that then if you look at the points in the set  $G = \{\dots, -\frac{2}{n}, -\frac{1}{n}, 0, \frac{1}{n}, \frac{2}{n}, \dots\}$  there will be one of these points between  $x$  and  $y$ . In fact we take the smallest point in  $G$  which is larger than  $x$ . However, to show that such a smallest point exists, we must eliminate two possibilities

- (1). It could be that none of the points in  $G$  is larger than  $x$ .
- (2). It could be that there is no smallest point in  $G$  larger than  $x$ , in other words, that we can find  $k$  arbitrarily small (negative) with  $\frac{k}{n} > x$ .

To eliminate these two possibilities we use the Archimedean property. First we pick  $m_1$  with  $nx < m_1$ . Then we pick  $m_2$  with  $-nx < m_2$ . Then

$$-\frac{m_2}{n} < x < \frac{m_1}{n}.$$

We thus know that if  $\frac{k}{n} > x$  then  $k > -m_2$ . Hence we can choose the smallest value  $k$  with  $\frac{k}{n} > x$  from among the finite set of possible values  $-m_2+1, -m_2+2, \dots, m_1$ .