Math 140A. Foundations of Analysis.
Text: Rudin. Instructor: Okikiolu

Lecture 4: Complex vectors.

We covered 1.25-1.38. However, we gave the following geometric proof of Theorem 1.35. We consider the space $\mathbb{C}^n$ of complex n-vectors, that is vectors of the form $(z_1, z_2, \ldots, z_n)$ where $z_1, \ldots, z_n$ are complex numbers. There is an **inner product** on $\mathbb{C}^n$ given by

$$\langle (z_1, \ldots, z_n), (w_1, \ldots, w_n) \rangle \;=\; \sum_{j=1}^{n} z_j \bar{w}_j.$$

Then this satisfies several properties
(a) $\langle \mathbf{z} + \mathbf{w}, \mathbf{u} \rangle \;=\; \langle \mathbf{z}, \mathbf{u} \rangle + \langle \mathbf{w}, \mathbf{u} \rangle$,
(b) $\langle \lambda \mathbf{z}, \mathbf{w} \rangle \;=\; \lambda \langle \mathbf{z}, \mathbf{w} \rangle$, and $\langle \mathbf{z}, \lambda \mathbf{w} \rangle \;=\; \bar{\lambda} \langle \mathbf{z}, \mathbf{w} \rangle$ for complex λ.
(c) $\langle \mathbf{z}, \mathbf{w} \rangle \;=\; \overline{\langle \mathbf{w}, \mathbf{z} \rangle}$.
(d) $\langle \mathbf{v}, \mathbf{v} \rangle \;\geq\; 0$, with equality if and only if $\mathbf{v}$ is the zero vector.

Using this we define $|\mathbf{v}| = \langle \mathbf{v}, \mathbf{v} \rangle^{1/2}$.

(e) $|\langle \mathbf{v}, \mathbf{w} \rangle| \;\leq\; |\mathbf{v}|\;|\mathbf{w}|$. (Cauchy Schwarz inequality)
(f) $|\mathbf{v} + \mathbf{w}| \;\leq\; |\mathbf{v}| + |\mathbf{w}|$. (triangle inequality)

We give the proofs of (e) and (f).

For (e), the inequality is trivial if $\langle \mathbf{v}, \mathbf{w} \rangle = 0$, so we can assume that this is not the case so that in particular $\mathbf{v}$ and $\mathbf{w}$ are both non-zero. We apply (d) to $\mathbf{v} - \lambda \mathbf{w}$ to get

$$0 \;\leq\; \langle \mathbf{v} - \lambda \mathbf{w}\,,\, \mathbf{v} - \lambda \mathbf{w} \rangle \;=\; \langle \mathbf{v}, \mathbf{v} \rangle \;-\; \lambda \langle \mathbf{w}, \mathbf{v} \rangle \;-\; \bar{\lambda} \langle \mathbf{v}, \mathbf{w} \rangle \;+\; |\lambda|^2 \langle \mathbf{w}, \mathbf{w} \rangle$$

Now we make the right hand side real by picking $\lambda = r \langle \mathbf{v}, \mathbf{w} \rangle$ where r is real. This gives

$$0 \;\leq\; \langle \mathbf{v}, \mathbf{v} \rangle \;-\; 2r |\langle \mathbf{v}, \mathbf{w} \rangle|^2 \;+\; r^2 |\langle \mathbf{v}, \mathbf{w} \rangle|^2 \langle \mathbf{w}, \mathbf{w} \rangle$$

Now choosing $r = 1/\langle \mathbf{w}, \mathbf{w} \rangle$, we get

$$0 \;\leq\; \langle \mathbf{v}, \mathbf{v} \rangle \;-\; \frac{|\langle \mathbf{v}, \mathbf{w} \rangle|^2}{\langle \mathbf{w}, \mathbf{w} \rangle}.$$

This gives

$$|\langle \mathbf{v}, \mathbf{w} \rangle|^2 \;\leq\; \langle \mathbf{v}, \mathbf{v} \rangle \; \langle \mathbf{w}, \mathbf{w} \rangle.$$

For (f), we use (e) to see that

$$\langle \mathbf{v}{+}\mathbf{w}\,,\, \mathbf{v}_{\mathbf{w}} \rangle \;=\; \langle \mathbf{v}, \mathbf{v} \rangle + \langle \mathbf{v}, \mathbf{w} \rangle + \langle \mathbf{w}, \mathbf{v} \rangle + \langle \mathbf{w}, \mathbf{w} \rangle \;\leq\; \langle \mathbf{v}, \mathbf{v} \rangle + 2|\mathbf{v}|\,|\mathbf{w}| + \langle \mathbf{w}, \mathbf{w} \rangle \;=\; (|\mathbf{v}|{+}|\mathbf{w}|)^2.$$