Math 140A. Foundations of Analysis.
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Lecture 5: Countable and Uncountable sets.

Let A and B be two sets. A **function** (or mapping) f from A to B is takes each element x of A and associates to it an element $f(x)$ in B.

A is called the **domain** of f.
The set $\{f(x) : x \in A\}$ is denoted by $f(A)$ and called the **range** of f. If $f(A) = B$ then f is called **onto**.

If $E \subset A$ then the set $\{f(x) : x \in E\}$ is denoted by $f(E)$ and called the *image* of E under f.

If $E \subset B$ then $\{x \in A : f(x) \in E\}$ is denoted by $f^{-1}(E)$ and is called the **inverse image** of E under f. In particular, if $y \in B$ then we define $f^{-1}(y) := f^{-1}(\{y\})$, which is the set of all $x \in A$ such that $f(x) = y$.

The function f is **1-1 (one-to-one)** if for every element $y \in B$ there is at most one element $x \in A$ with $f(x) = y$. Equivalently, if for each $y \in B$, the set $f^{-1}(y)$ contains at most one element. Equivalently if $f(x_1) = f(x_2)$ for some $x_1, x_2 \in A$, then $x_1 = x_2$.

Definition. Suppose A and B are two sets. We say that A and B can be put in 1-1 **correspondence**, or equivalently that A and B have the same **cardinality**, or that they are **equivalent** if there exists a one-to-one mapping from A onto B. We write $A \sim B$. The relation $\sim$ is

(a) reflexive: $A \sim A$.
(b) symmetric: $A \sim B$, then $B \sim A$.
(c) transitive: $A \sim B$, $B \sim C$, then $A \sim C$.

To see this, if $f : A \to B$ is 1-1 and onto, then $f^{-1} : B \to A$ is a function and is 1-1 and onto. If $f : A \to B$ and $g : B \to C$ are 1-1 and onto, then the composition $g \circ f : x \to g(f(x))$ is 1-1 and onto.

Examples. Decide which of the following sets are equivalent to each other:

$A = \{1, 2, 3, 4, 5\}$, $B = \{1, 2, 3\}$, $C = \{2, 4, 6\}$, $D = \{-1, 0, 2, 2, 2\}$,
$E = \{1, 2, 3, \dots\} = \{\text{positive integers}\}$, $F = \{2, 4, 6, \dots\} = \{\text{positive even integers}\}$,
$G = \{\dots, -1, 0, 1, 2, \dots\} = \{\text{integers}\}$, $H = \{\text{rationals in } [0, 1]\}$,
$J = \{(x, y) : x, y \in \{1, 2, 3, \dots\}\}$, $K = \{\text{rationals}\}$, $L = \{\text{real numbers}\}$.

Answers: A is not equivalent to any of the other sets.
$B \sim C \sim D$.
$E \sim F \sim G \sim H \sim J \sim K$.
L is not equivalent to any of the other sets.

We define $f : E \to F$ by $f(x) = 2x$.
We define $f : E \to G$ by

$$f(x) \;=\; \begin{cases} (x-1)/2 & x \text{ odd}, \\ -x/2 & x \text{ even} \end{cases}.$$

We define $f : E \to H$ by giving the sequence $x_1 = f(1),\ x_2 = f(2),\ x_3 = f(3),\ \ldots$ as follows:

$$0,\ 1,\ \frac{1}{2},\ \frac{1}{3},\ \frac{2}{3},\ \frac{1}{4},\ \frac{3}{4},\ \frac{1}{5},\ \frac{2}{5},\ \frac{3}{5},\ \frac{4}{5},\ \frac{1}{6},\ \frac{5}{6},\ \ldots$$

Every rational in $[0, 1]$ is listed exactly once here. Similarly, we list the elements in J as

$$(1,1),\ (1,2),\ (2,1),\ (1,3),\ (2,2),\ (3,1),\ \ldots$$

To see that $E \sim K$, if we first list the positive rationals as $x_1, x_2, \ldots,$ then we can list all the rationals as

$$0,\ x_1,\ -x_1,\ x_2,\ -x_2,\ \ldots$$

To list the positive rationals, we can first assign a rational to each element of J by sending

$$(j,k) \to k/j.$$

If

$$(j_1,k_1),\ (j_2,k_2),\ \ldots$$

is any listing of the set J, then

$$k_1/j_1,\ k_2/j_2,\ \ldots$$

is a listing of all the positive rationals. The only thing which remains to be done is to leave out repetitions.