MATH 140A. FOUNDATIONS OF ANALYSIS.
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Lecture 6. Countable and Uncountable sets.

Definition 2.4. Let A be a set.
(a) If $A \sim \{1, 2, \dots, n\}$ for some n, then A is finite.
(b) A is infinite if A is not finite.
(c) A is countable if $A \sim \{1, 2, 3, \dots\} = \{\text{positive integers}\}$.
(d) A is uncountable if A is neither finite nor countable.
(e) A is at most countable if A is finite or countable.

Remark. Many authors define a set to be countable if it is either finite or equivalent to the positive integers. While I prefer this definition because I like to think of finite sets as countable (their elements can after all be counted) we will stick with the notation of the book.

Lemma. *Suppose that E is an infinite set whose elements can be listed as x_1, x_2, $\dots$. Then E is countable.*

Proof. We just have to throw away repetitions from the sequence. To so this, we choose $i_1 < i_2 < \dots$ so that $i_1 = 1$, and if $i_1, \dots, i_{k-1}$ is chosen, we pick i_k to be the smallest number so that

$$x_{i_k} \notin \{x_1, x_2, \dots, x_{i_{k-1}}\}.$$

Note that one can then prove by induction that

$$x_{i_1}, x_{i_2}, x_{i_3}, \dots$$

is a sequence listing all elements of E exactly once. To show this, just use induction to show that

$$\{x_1, x_2, \dots, x_{i_k}\} = \{x_{i_1}, x_{i_2}, \dots, x_{i_k}\}.$$

This completes the proof started last time that the rationals are countable.

Definition. If A and Ω are sets and if for each $\alpha \in A$ the set E_α is a subset of Ω, then

$$\bigcup_{\alpha \in A} E_\alpha = \{x \in \Omega : x \in E_\alpha \text{ for at least one } \alpha \in A\}.$$

We call the next results "magic theorems" because for they can be used in most cases to establish that a countable set is indeed countable.

Magic Theorems. *1. An infinite subset of a countable set is countable.*
2. A countable union of countable sets is countable.

Proof of Theorems 1. If x_1, x_2, $\dots$ is a list of the elements of S which lists each element once, and if E is a subset of S, we can list the elements of E by setting

i_1 to be the smallest positive integer with $x_{i_1} \in E$, and if $i_1, \ldots, i_{k-1}$ have been chosen, then set i_k to be the smallest integer larger than i_{k-1} such that $x_{i_k} \in E$.

2. We already showed that the set of all elements (j,k) with j and k positive integers is countable, so we can list them as

$$(j_1, k_1),\ (j_2, k_2),\ \ldots$$

Now given a countable collection of sets E_1, E_2, $\ldots$ for which each E_k is countable, we can list the elements of E_j as

$$x_{j1},\ x_{j2},\ x_{j3},\ \ldots$$

Then we can list the elements of $\bigcup_{j=1}^{\infty} E_j$ as

$$x_{j_1 k_1},\ x_{j_2 k_2},\ x_{j_3 k_3},\ \ldots$$

This will repeat any element which occurs in more than one of the Ejs. However, by using the lemma we see that we can remove repetitions, so $\bigcup_{j=1}^{\infty} E_j$ is countable.

Theorem 2.14. *Let A be the set of all sequences whose elements are the digits 0 and 1. Then A is not countable.*

Proof. First we give examples of some elements in A.

$$(0,0,0,0,\ldots), \qquad (0,1,0,1,0,1,\ldots),$$

Suppose $\sqrt{2}$ has decimal expansion $b_0.b_1b_2b_3\ldots$, then another example of an element of A is $(a_1, a_2, \ldots)$" where

$$a_j = \begin{cases} 0 & \text{if } b_j \text{ is even,} \\ 1 & \text{if } b_j \text{ is odd.} \end{cases}$$

Now suppose that $a^{(1)}$, $a^{(2)}, \ldots$ is a list of all the elements of A and write

$$a^{(j)} \;=\; (a_1^{(j)},\ a_2^{(j)},\ \ldots).$$

Then define $(b_1, b_2, \ldots)$ by

$$b_j \;=\; \begin{cases} 0 & \text{if } a_j = 1 \\ 1 & \text{if } a_j = 0. \end{cases}$$

Since b differs from $a^{(j)}$ in the jth position, if is not equal to any of the elements $a^{(j)}$ in the list. Hence A cannot be countable.