Math 140B. Homework 2.

Reading: Rudin 94-97. 103-106.

H1. Show that if x_1 and y_1 are positive real numbers and if x_n, y_n are defined recursively by

$$\begin{pmatrix} x_{n+1} \\ y_{n+1} \end{pmatrix} = \begin{pmatrix} (x_n + y_n)/2 \\ \sqrt{x_n y_n} \end{pmatrix},$$

then the vector $\begin{pmatrix} x_n \\ y_n \end{pmatrix}$ converges in $\mathbb{R}^2$.

H2. Suppose that $f : (a, b) \to \mathbb{R}$ is a function and $c \in (a, b)$. Write down what it means for $f(c+)$ to exist. What can you say about f if you know that $f(c-) = f(c) = f(c+)$? How about if you just know that $f(c-) = f(c+)$?

H3. Give a function $f : [0, 1] \to \mathbb{R}$ which is discontinuous at uncountably many values in $[0, 1]$.

Definition. A function $f : [a, b] \to \mathbb{R}$ is said to be of **bounded variation** if there exists a constant C such that for every positive integer n, if we have any points $x_0, \ldots, x_n$ which satisfy $a = x_0 \leq x_1 \leq \cdots \leq x_n = b$, then

$$\sum_{k=1}^{n} |f(x_n) - f(x_{n-1})| \leq C.$$

H4. Show that if f is a function of bounded variation on $[a, b]$, and S is the set of points at which f is discontinuous, then S is at most countable.

Remark. If can be show that a function of bounded variation can be written as $u - v$ where u and v are both monotonically increasing functions.