MATH 140B HOMEWORK 4. KATE OKIKIOLU

Fact 1.

$$\frac{z^n - w^n}{z - w} = z^{n-1} + z^{n-2}w + \dots + zw^{n-2} + w^{n-1},$$

HW 1. *Assume that $|z| < R$ and $|w| < R$. (i) Show that*

$$\left| \frac{z^n - w^n}{z - w} \right| \leq n\, R^{n-1}.$$

(ii) Similarly, show that

$$\left| \frac{z^n - w^n}{z - w} - nw^n \right| \leq n(n-1)\, R^{n-2}|z - w|.$$

HW 2. *Suppose $|z| < R, |w| < R$. Suppose $a_0, a_1, \dots$ is a sequence of complex numbers with*

$$\sum_{n=2}^{\infty} n(n-1)|a_n|R^{n-2} < \infty.$$

Show that

$$f(z) = \sum_{n=0}^{\infty} a_n z^n, \qquad g(z) = \sum_{n=1}^{\infty} n a_n z^{n-1}$$

have radius of convergence at least R, and show that f is (complex) differentiable with derivative g on $|z| < R$.

HW 3. *Suppose that $f : \mathbb{C} \to \mathbb{C}$ is (complex) differentiable with $f' = 0$ on $\mathbb{C}$. Show that f is a constant. (Hint: deduce this from the real variable result.)*

Definition 1.

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}.$$

HW 4. *By considering the function $f(w) = \exp(z + w)\exp(-z)$, show that*

$$\exp(z + w) = \exp z \, \exp w.$$

Fact 2. *If $f : \mathbb{R} \to (0, \infty)$ is strictly increasing, continuous, and onto then f^{-1} : $(0, \infty) \to \mathbb{R}$ exists and is continuous. If f is differentiable, $f(x) = y$ and $f'(x) \neq 0$, then*

$$(f^{-1})'(y) = \frac{1}{f'(x)}.$$

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HW 5. *Show that*

$$\frac{d}{dy} \log y = \frac{1}{y},$$

and hence

$$\log(1 + t) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} t^n}{n}, \qquad |t| < 1$$