

## HOMEWORK 6, SOLUTIONS.

2. Suppose that  $\mathbf{N}$  is the unit normal to the surface,  $\mathbf{b}$  is the unit normal to the plane, and  $\alpha(s)$  is the curve parameterized by arclength (we assume that it is smooth or at least differentiable). Because the surface is normal to the plane along the curve we have

$$\mathbf{N}(\alpha(s)) = \pm \mathbf{b}.$$

Differentiating with respect to  $s$  we have

$$0 = \frac{d}{ds} \mathbf{N}(\alpha(s)) = d\mathbf{N}_{\alpha(s)}(\alpha'(s)).$$

Hence  $d\mathbf{N}_{\alpha(s)}$  has eigenvalue 0, and one of the principal curvatures vanishes. This means that the point  $\alpha(s)$  is either parabolic or planar.

5. The mean curvature is  $\frac{1}{2}(k_1 + k_2)$ . Now suppose that  $\mathbf{e}_1$  and  $\mathbf{e}_2$  are the principal directions which are orthogonal, and let  $\mathbf{e}$  be the “fixed direction”, which is a unit vector making an angle  $+\theta_0$  to  $\mathbf{e}_1$  (in the direction towards  $\mathbf{e}_2$ ), and suppose that  $\mathbf{w}$  makes an angle  $+\theta$  with  $\mathbf{e}$ . Then

$$\mathbf{w} = \cos(\theta + \theta_0)\mathbf{e}_1 + \sin(\theta + \theta_0)\mathbf{e}_2,$$

and by Euler’s formula,

$$k_n(\theta) = k_1 \cos^2(\theta + \theta_0) + k_2 \sin^2(\theta + \theta_0).$$

But then

$$\int_0^\pi k_n(\theta) d\theta = \int_0^\pi (k_1 \cos^2(\theta + \theta_0) + k_2 \sin^2(\theta + \theta_0)) d\theta = \frac{\pi}{2}(k_1 + k_2).$$

This gives the result.

7. Mean curvature zero implies  $k_2 = -k_1$ . By Euler’s formula, if  $\mathbf{w} = \cos \theta \mathbf{e}_1 + \sin \theta \mathbf{e}_2$  then the normal curvature in direction  $\mathbf{w}$  is

$$k_1(\cos^2 \theta - \sin^2 \theta).$$

This vanishes when  $\theta = \pi/4$  or  $\theta = 5\pi/4$ , which gives two orthogonal asymptotic directions.

8. (a). Either the lower hemisphere without the equator or the upper hemisphere without the equator, depending on your choice of normal.

(b). The sphere minus the poles.

(c). The sphere minus the poles.

9 (a). Firstly the composition  $N \circ \alpha$  is smooth. We have been assuming this without proof, but in fact we have two ways to prove this. One method is to note that you can “fatten”  $\mathbf{N}$  to a smooth map on an open neighborhood of each point  $\mathbf{p}$  of  $S$  by taking a parameterization  $\mathbf{r} : U \rightarrow S$  around  $\mathbf{p}$  and considering

$$\mathbf{N} = (\mathbf{N} \circ \mathbf{r}) \circ (\mathbf{r}^{-1}).$$

Since  $\mathbf{r}^{-1}$  can be fattened to a smooth map from a neighborhood of  $\mathbf{p}$ , and  $\mathbf{N} \circ \mathbf{r}$  is smooth on  $U$ , the composition  $\mathbf{N}$  is smooth.

It remains to prove that the tangent

$$(\mathbf{N} \circ \alpha)'$$

does not vanish. This is true because

$$(\mathbf{N} \circ \alpha)'(t) = d\mathbf{N}_{\alpha(t)}(\alpha'(t)).$$

Since  $\alpha$  is regular,  $\alpha'(t) \neq 0$ , and since  $S$  has no parabolic or planar points,  $d\mathbf{N}_{\alpha(t)}$  does not map any vector to zero.

(b).  $C = \alpha(I)$  is a *line of curvature* means that parameterizing  $\alpha$  by arclength  $s$ ,

$$(\mathbf{N} \circ \alpha)'(s) = k_n \alpha'(s)$$

where  $k_n$  is the normal curvature of  $\alpha$ . From this we see that since  $|\alpha'(s)| = 1$ , if  $\sigma$  is the arclength parameter for  $\mathbf{N} \circ \alpha$  then

$$\frac{d\sigma}{ds} = k_n(s)$$

and

$$\frac{d}{d\sigma}(\mathbf{N} \circ \alpha) = \frac{1}{k_n(s)} \frac{d}{ds}(\mathbf{N} \circ \alpha) = \alpha'(s)$$

and the curvature of  $\mathbf{N} \circ \alpha$  is

$$k_N = \left| \frac{d}{d\sigma} \alpha'(s) \right| = \left| \frac{1}{k_n(s)} \alpha''(s) \right| = \frac{k(s)}{k_n(s)}.$$

where  $k(s)$  is the curvature of  $\alpha$  at  $s$ .

13. Let  $\alpha(s)$  be the asymptotic curve parameterized by arclength. Then since it is asymptotic, the normal curvature in the direction  $\alpha'(s)$  vanishes. Since the normal curvature is

$$-\mathbf{N}(\alpha(s)) \cdot \alpha''(s),$$

where  $\mathbf{N}$  is the unit normal on the surface. This vanishes so

$$\mathbf{N} \cdot \kappa \mathbf{n} = 0$$

where  $\mathbf{t}, \mathbf{n}, \mathbf{b}$  is the Frenet frame of the curve  $\alpha$ . Since  $\kappa \neq 0$  we get

$$\mathbf{N} \cdot \mathbf{n} = 0.$$

Since also  $\mathbf{N} \cdot \mathbf{t} = 0$  we have  $\mathbf{N} = \pm \mathbf{b}$ . Now

$$d\mathbf{N}(\mathbf{t}) = (\mathbf{N} \circ \alpha)' = \mathbf{b}' = \tau \mathbf{n}.$$

But then since  $d\mathbf{N}$  is self-adjoint, the matrix of  $d\mathbf{N}$  in the basis  $\mathbf{t}, \mathbf{n}$  is

$$\begin{pmatrix} 0 & \tau \\ \tau & ? \end{pmatrix}$$

and

$$K = \det d\mathbf{N} = -\tau^2.$$

This is what we wanted to prove.