HOMEWORK 8, SELECTED SOLUTIONS SUMMARIZED.

§4-2 #1. F is clearly smooth. To check it is a *local* diffeomorphism we just need to check that dF is one-to-one.

$$F_u = (\sin\alpha\cos v, \sin\alpha\sin v, \cos\alpha), \qquad F_v = u\sin\alpha(-\sin v, \cos v, 0),$$

so

$$F_u \times F_v = u\sin\alpha(-\cos\alpha\cos v, -\cos\alpha\sin v, \sin\alpha), \qquad |F_u \times F_v| = u\sin\alpha$$

which is non-zero (provided $\sin\alpha \neq 0$), so dF is one-to-one. We have

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & u^2\sin^2\alpha \end{pmatrix}.$$

It is not a local isometry about any of its points because $u^2\sin^2\alpha$ is never locally identically equal to one.

§4-3 # 8b.

$$\mathbf{r}(r,\theta) = (r\cos\theta, r\sin\theta).$$

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}.$$

$$0 = \frac{1}{2}E_r = \langle \mathbf{r}_{rr}, \mathbf{r}_r \rangle = \Gamma^1_{11}E + \Gamma^2_{11}F = \Gamma^1_{11}.$$

$$0 = F_r - \frac{1}{2}E_\theta = \langle \mathbf{r}_{rr}, \mathbf{r}_\theta \rangle = \Gamma^1_{11}F + \Gamma^2_{11}G = \Gamma^2_{11}r^2.$$

$$0 = \frac{1}{2}E_\theta = \langle \mathbf{r}_{r\theta}, \mathbf{r}_r \rangle = \Gamma^1_{12}E + \Gamma^2_{12}F = \Gamma^1_{12}.$$

$$r = \frac{1}{2}G_r = \langle \mathbf{r}_{r\theta}, \mathbf{r}_\theta \rangle = \Gamma^1_{12}F + \Gamma^2_{12}G = \Gamma^2_{12}r^2.$$

$$-r = F_\theta - \frac{1}{2}G_r = \langle \mathbf{r}_{\theta\theta}, \mathbf{r}_r \rangle = \Gamma^1_{22}E + \Gamma^2_{22}F = \Gamma^1_{22}.$$

$$0 = \frac{1}{2}G_\theta = \langle \mathbf{r}_{\theta\theta}, \mathbf{r}_\theta \rangle = \Gamma^1_{22}F + \Gamma^2_{22}G = \Gamma^2_{22}r^2.$$

So $\Gamma^2_{12} = 1/r$, and $\Gamma^1_{22} = -r$.