LECTURE 1: REGULAR CURVES IN $\mathbb{R}^3$

September 21, 2001

Definition. The word smooth means infinitely differentiable.

Definition. A *smooth parameterized curve* $\mathbf{r}$ in $\mathbb{R}^3$ is a smooth function from an interval (a,b) into $\mathbb{R}^3$. (What it means for the function $\mathbf{r}$ into R^3 to be smooth is that the coordinate functions $x(t), y(t), z(t)$ given by

$$\mathbf{r}(t) = (x(t), y(t), z(t))$$

are smooth.)

Definition. The *trace* of $\mathbf{r}$ is its image $\mathbf{r}((a,b))$.

Definition. The tangent to the curve at t is the vector

$$\mathbf{r}'(t) = (x'(t), y'(t), z'(t))$$

(It is sometimes useful to think of the curve as the path of a moving particle and the parameter as representing time. The tangent is then the velocity.)

Definition. The smooth parameterized curve is called *regular* if the tangent is never zero. In this case the trace of the curve "looks nice and smooth".

Example 1. The plane curve

$$\mathbf{r}(t) = (t^2, t^3), \qquad t \in (-\infty, \infty)$$

does not "look nice and smooth" at $t = 0$.

Example 2. The curve may not be one-to-one as in the case

$$\mathbf{r}(t) = (\sin t, \sin 2t), \qquad t \in (0, \infty).$$

Definition. The parameterized curve $\mathbf{r}_2 : (a_2, b_2) \to \mathbb{R}^3$ is a **reparameterization** of the parameterized curve $\mathbf{r}_1 : (a_1, b_1) \to \mathbb{R}^3$ if there exists an increasing smooth function

$$T : (a_2, b_2) \to (a_1, b_1)$$

Typeset by $\mathcal{AMS}$-TEX

such that

$$\mathbf{r}_2(t) \;=\; \mathbf{r}_1(T(t)).$$

Example 3. The curve

$$(\sin t^2, \cos t^2), \qquad\qquad t \in (0, \infty)$$

is a reparameterization of the curve in Example 2.
A reparameterization has the same trace and *orientation* (*direction*) as the original curve. If $\mathbf{r} : (a, b) \to \mathbb{R}^3$ is a smooth curve, then the curve $\mathbf{r}_- =: (-b, -a) \to \mathbb{R}^3$ defined by

$$\mathbf{r}_-(t) \;=\; \mathbf{r}(-t)$$

is a curve with the same trace but opposite orientation.

Basic Problem. Suppose that we are given the trace of a curve in $\mathbb{R}^3$. What is the most efficient way to describe its geometric shape?

We are looking for something better than a general parameterization of the curve. Firstly, there are an infinite number of ways to parameterize the curve so the general parameterization is not unique. Secondly you need to give all 3 coordinate functions to describe the curve, whereas in fact only two functions are really needed. These shortcomings can be overcome by parameterizing by *arclength.* However to explain the *shape* of the curve at a given point you need to compute higher derivatives of the coordinate functions. Furthermore, if you choose a new origin and new axes in Euclidean space $\mathbb{R}^3$, it is not immediately obvious that the curve written in the new coordinates is the same as the old one. An alternative way of looking at this is if you rotate and translate the curve in the space, it retains its 'shape', but this is not immediately obvious from the equations.

We are going to describe the trace of a curve by giving a point on it, the unit tangent at that point, and two functions:

1. The 'curvature' of the curve at distance s along the curve from $\mathbf{r}_0$. This measures how tightly the trace is curving.
2. The 'torsion' of the curve at distance s along the curve from $\mathbf{r}_0$. This measures how close the trace is to lying in a plane.

First recall the notion of *arclength.* Consider the smooth curve $\mathbf{r} : (a, b) \;\to\; \mathbb{R}^3$. Suppose $t_0 \in)a, b)$ and set

$$s(t) \;=\; \int_{t_0}^{t} |\mathbf{r}'(\tau)|\, d\tau \;=\; \int_{a}^{b} \sqrt{x'(\tau)^2 + y'(\tau)^2 + z'(\tau)^2}\, d\tau.$$

This is the (signed) arclength along the curve from t_0 to t.

A curve $\mathbf{r}(t)$ is said to be *parameterized by arclength* if

$$|\mathbf{r}'(t)| = 1, \qquad \text{for all } t, \qquad\qquad \text{i.e. } s = t - t_0.$$

A regular curve $\mathbf{r}(t)$ can always be reparameterized by arclength. Indeed, solving for t in terms of s gives a smooth function since ds/dt is non-zero.
Recall the *scalar product* and *vector products*. If $\mathbf{a} = (a_1, a_2, a_3)$ and $\mathbf{b} = (b_1, b_2, b_3)$, then the scalar product is

$$\mathbf{a} \cdot \mathbf{b} \;=\; a_1 b_1 + a_2 b_2 + a_3 b_3 \;=\; |\mathbf{a}||\mathbf{b}| \cos\theta$$

where θ is the angle between $\mathbf{a}$ and $\mathbf{b}$. In particular $\mathbf{a} \cdot \mathbf{b} = 0$ if and only if $\mathbf{a}$ and $\mathbf{b}$ are perpendicular. Set

$$\mathbf{e}_1 = (1, 0, 0), \qquad \mathbf{e}_2 = (0, 1, 0), \qquad \mathbf{e}_3 = (0, 0, 1).$$

Then the vector product of $\mathbf{a}$ and $\mathbf{b}$ is

$$\mathbf{a} \times \mathbf{b} \;=\; \begin{vmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \mathbf{e}_3 \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \;=\; \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \mathbf{e}_1 \;-\; \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \mathbf{e}_2 \;+\; \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \mathbf{e}_3,$$

where the 2×2 determinant is given by

$$\begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \;=\; a_1 b_2 - a_2 b_1.$$

Then $\mathbf{a} \times \mathbf{b}$ is perpendicular to both $\mathbf{a}$ and $\mathbf{b}$, its length is the area of the parallelogram spanned by $\mathbf{a}$ and $\mathbf{b}$, that is

$$|\mathbf{a} \times \mathbf{b}| \;=\; |\mathbf{a}||\mathbf{b}| \sin\theta.$$

Furthermore, the triple $\mathbf{a}, \mathbf{b}, \mathbf{a} \times \mathbf{b}$ obey the right hand rule. In particular, $\mathbf{a} \times \mathbf{b} = \mathbf{O}$ if and only if $\mathbf{a}$ and $\mathbf{b}$ are parallel (or one is zero).

If $\mathbf{a}$ and $\mathbf{b}$ are smooth functions from (a, b) into $\mathbb{R}^3$ then we have the Leibnitz rules

$$(\mathbf{a} \cdot \mathbf{b})'(t) \;=\; \mathbf{a}'(t) \cdot \mathbf{b}(t) \;+\; \mathbf{a}(t) \cdot \mathbf{b}'(t)$$

and

$$(\mathbf{a} \times \mathbf{b})'(t) \;=\; \mathbf{a}'(t) \times \mathbf{b}(t) \;+\; \mathbf{a}(t) \times \mathbf{b}'(t)$$

Curvature. From now on we consider a general regular curve $\mathbf{r}(s)$ parametermized by arclength. Then

$$\mathbf{t}(s) \;=\; \mathbf{r}'(s)$$

is a unit tangent vector. The *curvature* of the curve at s is

$$\kappa \;=\; |\mathbf{t}'(s)| \;=\; |\mathbf{r}''(s)|.$$

The more tightly the curve is curving, the larger the curvature.
Lemma. The curvature of the circle of radius ρ is $1/\rho$.
Proof. Arclength parameterization of the circle radius ρ in the place with center $(0,0)$ is given by

$$(\rho \cos \tfrac{s}{\rho}, \rho \sin \tfrac{s}{\rho}).$$

Then

$$\mathbf{t} \;=\; (-\sin \tfrac{s}{\rho}, \cos \tfrac{s}{\rho})$$

and

$$\mathbf{t}' \;=\; (-(1/\rho)\cos \tfrac{s}{\rho}, -(1/\rho)\sin \tfrac{s}{\rho}),$$

so $|\mathbf{t}'| = 1/\rho$.
In general $1/\rho(s)$ is called the *radius of curvature* of the curve at s. It is the radius of the circle which best approximates the curve at that point.

Lemma. $\mathbf{t}'(s)$ is perpendicular to $\mathbf{t}(s)$. (If a particle is traveling with constant speed then its acceleration is perpendicular to its motion!)
Proof.

$$\mathbf{t}\cdot\mathbf{t} \;=\; 1$$

so

$$0 \;=\; (\mathbf{t}\cdot\mathbf{t})' \;=\; \mathbf{t}'\cdot\mathbf{t} \;+\; \mathbf{t}\cdot\mathbf{t}' \;=\; 2\mathbf{t}'\cdot\mathbf{t}.$$