LECTURE 10: THE DIFFERENTIAL OF A MAP OF A REGULAR SURFACE.

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New Format: When we have followed Do Carmo closely in the lecture or when we have reviewed material from the 20E syllabus, we will only summarize the results and definitions here and omit the proofs.

Lemma (20E). $dF_{\mathbf{p}}$ is normal to the surface $F = a$ at $\mathbf{p}$.

Lemma (20E). The angle between two surfaces at a point where they intersect is the angle between their normals at the point.

Example: Calculate the angle between the surface $z = x^2 + y^2 - 1$ and the surface $z = 1 - x^2 - y^2$ at $(0, 1, 0)$.

Solution: The surfaces can be written $x^2 + y^2 - z = 1$ and $x^2 + y^2 + z = 1$, so the normals are $(2x, 2y, -1)$ and $(2x, 2y, 1)$. At $(0, 1, 0)$ these are $(0, 2, -1)$ and $(0, 2, 1)$. If θ is the angle between these normals, then

$$\cos\theta = \frac{(0, 2, -1) \cdot (0, 2, 1)}{|(0, 2, -1)||(0, 2, 1)|} = \frac{3}{5}.$$

Definition. Suppose that $F : U \to \mathbb{R}^m$ is a map where U is open in $\mathbb{R}^n$, and F is given by

$$F(x_1, \dots, x_n) = (F_1(x_1, \dots, x_n), \dots, F_m(x_1, \dots, x_n)),$$

and is smooth - that is the functions $F_1, \dots, F_m$ have partial derivatives of all orders. Let $p = (p_1, \dots, p_n) \in U$ and let $w = (w_1, \dots, w_n) \in \mathbb{R}^n$. The *directional derivative of F in direction w at p* is

$$dF_p w \;=\; \left.\frac{d}{dt}\right|_{t=0} F(p + tw).$$

When $w = e_i = (0, \dots, 0, 1, 0, \dots, 0)$ (where the '1' is in the i^{th} spot), this is just the partial derivative

$$dF_p e_i \;=\; \frac{\partial F}{\partial x_i}(p).$$

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For general w, the chain rule gives

$$\begin{aligned} dF_p w &= \frac{d}{dt}\bigg|_{t=0} F(p+tw) = \frac{\partial F}{\partial x_1}\bigg|_{x=p} \frac{d(p_1+tw_1)}{dt}\bigg|_{t=0} + \cdots + \frac{\partial F}{\partial x_n}\bigg|_{x=p} \frac{d(p_n+tw_n)}{dt}\bigg|_{t=0} \\ &= \begin{pmatrix} \frac{\partial F_1}{\partial x_1}(p) & \dots & \frac{\partial F_1}{\partial x_n}(p) \\ \vdots & \ddots & \vdots \\ \frac{\partial F_m}{\partial x_1}(p) & \dots & \frac{\partial F_m}{\partial x_n}(p) \end{pmatrix} \begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}, \end{aligned}$$

where in the last line we are assuming that the vector $DF_p w$ is written as a column rather than a row. We see that the map

$$w \to dF_p w$$

is a linear map dF_p. It can be shown that

$$\frac{|F(p+w) - F(p) - dF_p w|}{|w|} \to 0, \qquad \text{as } w \to 0.$$

We define the differential of a smooth map from $\mathbb{R}^m$ to $\mathbb{R}^n$ in terms of directional derivatives and as a linear map.

Example. Calculate $dF_{\mathbf{p}}(\mathbf{w})$ where $F : \mathbb{R}^3 \to \mathbb{R}^3$ is defined by

$$F(x,y,z) = (x-y, y^2, z),$$

$\mathbf{p} = (1,2,0)$ and $\mathbf{w} = (0,-1,1)$.

Solution. Calculating partial derivatives, the matrix of dF is

$$\begin{pmatrix} 1 & -1 & 0 \\ -1 & 2y & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

and multiplying this evaluated at $\mathbf{p} = (1,2,0)$ by $\mathbf{w}$ gives

$$dF_{\mathbf{p}}\mathbf{w} = \begin{pmatrix} 1 & -1 & 0 \\ -1 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ -4 \\ 1 \end{pmatrix}.$$

Definition. If S_1 and S_2 are regular surfaces, $F : S_1 \to S_2$ is smooth, $\mathbf{p}_1 \in S_1$, $\mathbf{p}_2 = F(\mathbf{p}_1)$ and $\mathbf{w} \in T_{\mathbf{p}_1} S_1$, we choose a smooth curve $\alpha : (-\varepsilon, \varepsilon) \to S_1$ with $\alpha(0) = \mathbf{p}_1$ and $\alpha'(0) = \mathbf{w}$. Then we define

$$dF_{\mathbf{p}_1}\mathbf{w} = \frac{d}{dt}\bigg|_{t=0} F(\alpha(t)).$$

As James remarked, we could have defined this by showing that we can "fatten up" F to a smooth map from a neighborhood U of $\mathbf{p}_1$ in $\mathbb{R}^3$. Then we could have used the definition given for differentiating smooth maps on $\mathbb{R}^3$. This procedure works. The second Lemma below shows that it gives the same definition.

Lemma (Do Carmo). The derivative $dF_{\mathbf{p}_1}\mathbf{w}$ just defined is well defined and gives a linear map $dF_{\mathbf{p}_1}$ from $T_{\mathbf{p}_1}S_1$ to $T_{\mathbf{p}_2}S_2$.

Example. The cylinder S can be parameterized by

$$\mathbf{r} : (\theta, z) \to (\cos\theta, \sin\theta, z), \qquad\qquad -\pi < \theta < \pi.$$

Let $F : S \to S$ be given by

$$F : \mathbf{r}(\theta, z) \ \to \ \mathbf{r}(2\theta, -z)$$

on a neighborhood of $(\theta, z) = (0, 0)$. Calculate $dF(0, 1, -1)$ at $(1, 0, 0)$.

Solution. Note that F maps $\mathbf{r}(0, 0) = (1, 0, 0)$ to $\mathbf{r}(0, 0) = (1, 0, 0)$. In the parameterization $\mathbf{r}$ the matrix of dF is

$$\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}.$$

We must express the vector $(0, 1, -1)$ in the basis associated to $\mathbf{r}$. This basis is

$$\frac{\partial \mathbf{r}}{\partial \theta} = (-\sin\theta, \cos\theta, 0), \qquad \frac{\partial \mathbf{r}}{\partial z} = (0, 0, 1).$$

At $(0, 0)$ this is

$$(0, 1, 0), \qquad (0, 0, 1).$$

We easily find that

$$(0, 1, -1) \ = \ 1(0, 1, 0) \ + \ -1(0, 1, 0)$$

so $(0, 1, -1)$ has coordinates $(1, -1)$. Then $dF_{(1,0,0)}(0, 1, -1)$ has coordinates

$$\begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}.$$

This is again at $(\theta, z) = (0, 0)$ where we have already computed the basis associated to $\mathbf{r}$. We have

$$dF_{(1,0,0)}(0, 1, -1) = 2(0, 1, 0) + 1(0, 0, 1) = (0, 2, 1).$$

Lemma. If $F : U \to \mathbb{R}^3$ is smooth and maps the regular surface S_1 to S_2 and $\mathbf{p} \in S_1$ then for $\mathbf{w} \in T_{\mathbf{p}}\mathbf{w}$,

$$d(F|_{S_1})_{\mathbf{p}}\mathbf{w} \ = \ dF_{\mathbf{p}}\mathbf{w}.$$

Example. In the previous example,

$$F(\cos\theta, \sin\theta, z) = (\cos 2\theta, \sin 2\theta, -z) = (\cos^2\theta - \sin^2\theta, 2\cos\theta\sin\theta, -z).$$

We can extend F from the cylinder to $\mathbb{R}^3$ by setting

$$F(x, y, z) = (x^2 - y^2, 2xy, -z).$$

Then dF has matrix

$$\begin{pmatrix} 2x & -2y & 0 \\ 2y & 2x & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

and at $(1, 0, 0)$,

$$\begin{pmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 1 \end{pmatrix}.$$