LECTURE 11: THE FIRST FUNDAMENTAL FORM.

October 29, 2001

New Format: When we have followed Do Carmo closely in the lecture or when we have reviewed material from the 20E syllabus, we will only summarize the results and definitions here and omit the proofs.

Today's Quiz. What is a vector space?
Recall: Last week we showed that the space of tangent vectors to a regular surface at a given point forms a plane - that is a 2-dimensional vector space.

Let V be a plane in $\mathbb{R}^3$ (or more generally any vector space). A *bilinear form* on V is a map

$$B : V \times V \to \mathbb{R}$$

satisfying

$$\begin{aligned} B(\mathbf{u}+\mathbf{v}+\mathbf{w}) &= B(\mathbf{u},\mathbf{w}) + B(\mathbf{v},\mathbf{w}), & B(\lambda\mathbf{u},\mathbf{v}) &= \lambda B(\mathbf{u},\mathbf{v}) \\ B(\mathbf{u},\mathbf{v}+\mathbf{w}) &= B(\mathbf{u},\mathbf{v}) + B(\mathbf{u},\mathbf{w}), & B(\mathbf{u},\lambda\mathbf{v}) &= \lambda B(\mathbf{u},\mathbf{v}). \end{aligned}$$

The most familiar example is the scalar product on $\mathbb{R}^2$

$$(*) \qquad B((a_1,b_1),(a_2,b_2)) = (a_1,b_1)\cdot(a_2,b_2) = a_1a_2 + b_1b_2.$$

This example is *symmetric* and *positive definite.* B is symmetric if for all $\mathbf{u},\mathbf{v}$,

$$B(\mathbf{u},\mathbf{v}) = B(\mathbf{v},\mathbf{u}),$$

and is positive definite if

$$B(\mathbf{u},\mathbf{u}) \geq 0, \qquad B(\mathbf{u},\mathbf{u}) = 0 \Leftrightarrow \mathbf{u} = 0.$$

A symmetric positive definite bilinear form is called an *inner product.* The general symmetric bilinear form on $\mathbb{R}^2$ has the form

$$\begin{aligned} B((a_1,b_1),(a_2,b_2)) &= a_1a_2B(\mathbf{e}_1,\mathbf{e}_1) + a_1b_2B(\mathbf{e}_1,\mathbf{e}_2) \\ &\quad + b_1a_2B(\mathbf{e}_2,\mathbf{e}_1) + b_1b_2B(\mathbf{e}_1,\mathbf{e}_2) \\ &= (a_1,b_1)\begin{pmatrix} B(\mathbf{e}_1,\mathbf{e}_1) & B(\mathbf{e}_1,\mathbf{e}_2) \\ B(\mathbf{e}_2,\mathbf{e}_1) & B(\mathbf{e}_2,\mathbf{e}_2) \end{pmatrix}\begin{pmatrix} a_2 \\ b_2 \end{pmatrix} \end{aligned}$$

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The *matrix* of the bilinear form is

$$\begin{pmatrix} E & F \\ H & G \end{pmatrix} = \begin{pmatrix} B(\mathbf{e}_1, \mathbf{e}_1) & B(\mathbf{e}_1, \mathbf{e}_2) \\ B(\mathbf{e}_2, \mathbf{e}_1) & B(\mathbf{e}_2, \mathbf{e}_2) \end{pmatrix}.$$

Now suppose that V is a general plane in $\mathbb{R}^3$. The scalar product on $\mathbb{R}^3$ restricts to the plane to give an inner product $\langle\, , \,\rangle$. If we are given vectors $\mathbf{w}_1$ and $\mathbf{w}_2$ spanning the plane, then writing vectors in the plane in their coordinates with respect to $\mathbf{w}_1$ and $\mathbf{w}_2$ we get the form (**). Indeed, repeating the previous calculation,

$$\begin{aligned} \langle \mathbf{a}_1, \mathbf{a}_2 \rangle &= \langle a_1\mathbf{w}_1 + b_1\mathbf{w}_2, a_2\mathbf{w}_1 + b_2\mathbf{w}_2 \rangle \\ &= a_1a_2\langle \mathbf{w}_1, \mathbf{w}_1\rangle \ + \ (a_1b_2 + a_2b_1)\langle \mathbf{w}_1, \mathbf{w}_2\rangle \ + \ b_1b_2\langle \mathbf{w}_2, \mathbf{w}_2\rangle \\ (***) \qquad &= (a_1, b_1) \begin{pmatrix} \langle \mathbf{w}_1, \mathbf{w}_1\rangle & \langle \mathbf{w}_1, \mathbf{w}_2\rangle \\ \langle \mathbf{w}_2, \mathbf{w}_1\rangle & \langle \mathbf{w}_2, \mathbf{w}_2\rangle \end{pmatrix} \begin{pmatrix} a_2 \\ b_2 \end{pmatrix} \end{aligned}$$

We set

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} \langle \mathbf{w}_1, \mathbf{w}_1\rangle & \langle \mathbf{w}_1, \mathbf{w}_2\rangle \\ \langle \mathbf{w}_2, \mathbf{w}_1\rangle & \langle \mathbf{w}_2, \mathbf{w}_2\rangle \end{pmatrix}.$$

We get for $\mathbf{a} = (\alpha, \beta, \gamma) = a\mathbf{w}_1 + b\mathbf{w}_2$,

$$|\mathbf{a}|^2 \ = \ \mathbf{a} \cdot \mathbf{a} \ = \ Ea^2 + 2Fab + Gb^2.$$

Remark: In the case when $\mathbf{w}_1$ and $\mathbf{w}_2$ are orthonormal (orthogonal and unit length), $E = 1$, $F = 0$, $G = 1$.

Example. Calculate E, F and G when

$$\mathbf{w}_1 = (2, 4, 0), \qquad \mathbf{w}_2 = (1, 1, -1).$$

Use this to calculate the length of the vector $\mathbf{a} = 2\mathbf{w}_1 - \mathbf{w}_2$.

Solution.

$$E = 2^2 + 4^2 = 20, \qquad F = 2 \cdot 1 + 4 \cdot 1 = 6, \qquad G = 1^2 + 1^2 + (-1)^2 = 3.$$

$$|2\mathbf{w}_1 - \mathbf{w}_2|^2 = (2, -1) \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} 2 \\ -1 \end{pmatrix} = 2^2E + 2(-2)F + (-1)^2G = 80 - 24 + 3 = 59.$$

Check: $\mathbf{a} = 2\mathbf{w}_1 - \mathbf{w}_2 = (3, 7, 1)$. $3^2 + 7^2 + 1^2 = 59$.

Now we can calculate lengths of curves defined on surfaces by using a parameterization. Indeed, let S be a regular surface and let $\mathbf{r} : U \to S$ be a parameterization of S. At each point $\mathbf{p}$ of S we write $\langle\, , \,\rangle$ for the scalar product coming from $\mathbb{R}^3$ restricted to

$T_{\mathbf{p}}S$. Sometimes we denote it by $\langle\, , \,\rangle_{\mathbf{p}}$ to avoid confusion. It is also denoted by $I(\, , \,)$ and is called the *first fundamental form.* Now $\partial\mathbf{r}/\partial u$ and $\partial\mathbf{r}/\partial v$ at (u, v) form a basis for $T_{\mathbf{r}(u,v)}S$. Set

$$E(u,v) = \left\langle \frac{\partial\mathbf{r}}{\partial u}(u,v), \frac{\partial\mathbf{r}}{\partial u}(u,v) \right\rangle, \qquad F(u,v) = \left\langle \frac{\partial\mathbf{r}}{\partial u}(u,v), \frac{\partial\mathbf{r}}{\partial v}(u,v) \right\rangle$$

$$G(u,v) = \left\langle \frac{\partial\mathbf{r}}{\partial v}(u,v), \frac{\partial\mathbf{r}}{\partial v}(u,v) \right\rangle.$$

Then

$$\langle d\mathbf{r}(a_1, b_1), d\mathbf{r}(a_2, b_2)\rangle = (a_1, b_1) \begin{pmatrix} E & F \\ F & G \end{pmatrix} \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}.$$

Example. Consider the parameterization of the paraboloid $z = x^2 + y^2$:

$$(x, y) \rightarrow (x, y, x^2 + y^2).$$

Calculate E, F, G and hence calculate the angle between $d\mathbf{r}(1, -1)$ and $d\mathbf{r}(1, 1)$ at $\mathbf{r}(2, 1)$.
Solution

$$\frac{\partial\mathbf{r}}{\partial x} = (1, 0, 2x), \qquad \frac{\partial\mathbf{r}}{\partial y} = (0, 1, 2y).$$

$$E = 1 + 4x^2, \qquad F = 4xy \qquad G = 1 + 4y^2.$$

At $(2, 1)$,

$$E = 17, \qquad F = 8, \qquad G = 5,$$

so

$$\langle \mathbf{r}(1,-1), \mathbf{r}(1,1)\rangle = (1,-1)\begin{pmatrix} 17 & 8 \\ 8 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = (1,-1)\begin{pmatrix} 25 \\ 13 \end{pmatrix} = 12$$

$$\langle \mathbf{r}(1,-1), \mathbf{r}(1,-1)\rangle = (1,-1)\begin{pmatrix} 17 & 8 \\ 8 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ -1 \end{pmatrix} = (1,-1)\begin{pmatrix} 9 \\ 3 \end{pmatrix} = 6$$

$$\langle \mathbf{r}(1,1), \mathbf{r}(1,1)\rangle = (1,1)\begin{pmatrix} 17 & 8 \\ 8 & 5 \end{pmatrix}\begin{pmatrix} 1 \\ 1 \end{pmatrix} = (1,1)\begin{pmatrix} 25 \\ 13 \end{pmatrix} = 48$$

Then if θ is the angle between the vectors,

$$\cos\theta = \frac{12}{\sqrt{6}\sqrt{48}} = \frac{1}{2\sqrt{2}}.$$

Examples from Do Carmo. For the parameterization of the cylinder

$$\mathbf{r}(\theta, z) = (\cos\theta, \sin\theta, z)$$

we compute

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

We'll see that this implies the cylinder is *locally isometric* to the plane.
For the parameterization of the sphere

$$\mathbf{r}(\phi, \theta) = (\sin\phi\cos\theta, \sin\phi\sin\theta, \cos\phi)$$

we compute

$$\begin{pmatrix} E & F \\ F & G \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & \sin^2\phi \end{pmatrix}.$$

We show that the curves which make a constant angle β with the lines of longitude θ constant, are given by

$$\log\tan\frac{\phi}{2} = \pm(\theta + c)\cot\beta.$$